

ON ARHANGELSKII'S PROBLEM
SH668

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ABSTRACT. We prove the consistency (modulo supercompact) of a negative answer to Arhangel'skii's problem (some Hausdorff compact space cannot be partitioned to two sets not containing a closed copy of Cantor discontinuum). In this model we have CH. Without CH we get consistency results using a pcf assumption, close relatives of which are necessary for such results.

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ANOTATED CONTENT

§1 General spaces: consistency from strong assumptions

[We define $X^* \rightarrow (Y^*)_\theta^1$ for topological spaces X^*, Y^* . Then starting with a Hausdorff space Y^* with θ points such that any set of $< \sigma$ members is discrete and $\kappa = \kappa^{<\kappa} \in (\theta, \lambda)$ and appropriate $\mathcal{A} \subseteq [\lambda]^\theta$ such that any two members has intersection $< \sigma$, we force appropriate X^* . We then show that the assumption holds under appropriate pcf assumption and finish with some improvements.]

§2 Consistency from supercompact, with clopen basis

[We deal here with the set theoretic assumption. We show that the assumptions can be gotten from supercompact for the case we agree to have CH, relying on earlier consistency results.]

§3 Equi-consistency

[We show that some versions of the topological question and suitable combinatorial questions are equi-consistent. See [Sh 108], [HJSh 249], [Sh 460], [Sh:F276]. Saharon We then indicate the changes needed for the not necessarily closed subspace case colouring by more colours and other spaces. For discussion see [Sh 666], §1.]

§4 Helping equi-consistency

§1 GENERAL SPACES: CONSISTENCY FROM STRONG ASSUMPTIONS

In our main theorem, 1.2, we give set theoretic sufficient conditions for being able to force counterexamples to Arhangel'skii's problem, possibly replacing the cantor discontinuum by any other space. It has a version for spaces with clopen basis. Then (in claim 1.4) we connect this to pcf theory: after easy forcing the assumptions of Theorem 1.2 can be proved, if we start with a suitable (strong) pcf assumption (whose status is not known). Then in claim 1.5 we deal with variants of the theorem, weakening the topological and/or set theoretic assumptions. Further variants are discussed in the end (T_3 spaces without clopen basis and variants of 1.4).

1.1 Definition. Let $n \in [1, \omega)$ (though we concentrate on $n = 1$) .

1) We say $X^* \rightarrow (Y^*)_\theta^n$, if X^*, Y^* are topological spaces and for every $h : [X^*]^n \rightarrow \theta$ there is a closed subspace Y of X^* , homeomorphic to Y^* such that $h \upharpoonright [Y]^n$ is constant (if $n = 1$ we may write $h : X^* \rightarrow \theta, h \upharpoonright Y$).

2) If we omit the "closed", we shall write \rightarrow_w instead of \rightarrow . We write $(Y^*)_{<\theta}^n$ meaning: for every $h : [X^*]^n \rightarrow \gamma < \theta$. We use $\nrightarrow, \nrightarrow_w$ for the negations.

1.2 Theorem. Assume

- (A) (i) $\lambda > \kappa > \theta > \sigma \geq \aleph_0$ and $\kappa = \kappa^{<\kappa}$
(ii) $(\forall \alpha < \kappa)(|\alpha|^\sigma < \kappa)$ and $\kappa > \theta^* \geq \theta$

- (B)₁ $\mathcal{A} \subseteq [\lambda]^\theta$ and
 $A_1 \neq A_2 \in \mathcal{A} \Rightarrow |A_1 \cap A_2| < \sigma$

- (B)₂ \mathcal{A} is $(< \kappa)$ -free which means: if $\mathcal{A}' \subseteq \mathcal{A}, |\mathcal{A}'| < \kappa$ then for some list $\{A_\varepsilon : \varepsilon < \zeta\}$ of \mathcal{A}' , for each $\varepsilon < \zeta$ we have
 $|A_\varepsilon \cap \bigcup_{\xi < \varepsilon} A_\xi| < \sigma$

- (C) if $F : \lambda \rightarrow [\lambda]^{\leq \kappa}$, then some $A \in \mathcal{A}$ (or just some A such that $(\exists A')(A \subseteq A' \text{ \& } |A| = \theta \text{ \& } A' \in \mathcal{A})$ is F -free which means

- (*) for $\alpha \neq \beta$ from A we have $\alpha \notin F(\beta)$

- (D) Y^* is a Hausdorff space with set of points θ and a basis $\mathcal{B} = \{b_i : i < \theta^*\}$

- (E) if Y is a subset of Y^* with $< \sigma$ points, then Y is a discrete subset (if $\sigma = \aleph_0$ this follows from Hausdorff), i.e. there is a sequence of open (for Y^*) pairwise disjoint sets $\langle \mathcal{U}_y : y \in Y \rangle$, such that $y \in \mathcal{U}_y$.

Then

1) for some κ -complete κ^+ -c.c. forcing notion P , in V^P there is X^* such that:

- (a) X^* is a Hausdorff topological space with λ points and basis of size $|\mathcal{A}| + \theta^*$
(b) $X^* \rightarrow (Y^*)_{<\text{cf}(\theta)}^1$ (that is if $X^* = \bigcup_{i < i(*)} X_i$ where $i(*) < \text{cf}(\theta)$ then some closed subspace Y of X^* homeomorphic to Y^* is included in some single X_i (i.e. $(\exists i)(Y \subseteq X_i)$).

2) If in addition Y^* has a clopen basis \mathcal{B} of cardinality $\leq \theta^*$ such that the union of $< \sigma$ members of \mathcal{B} is clopen, then we can require that X^* has a clopen basis.

1.3 Remark. We may define the conditions historically (see [ShSt 258], [RoSh 599], so put only the required conditions). Then we can allow $\theta^* = \kappa$, but see 1.5.

Proof. We write the proof for part (1) and indicate the changes for part (2). Without loss of generality

$$\otimes_1 (\forall \alpha < \beta < \lambda)(\forall B \in [\lambda]^{<\lambda})(\exists^{\kappa^+} A \in \mathcal{A})[\{\alpha, \beta\} \subseteq A \ \& \ A \cap B \subseteq \{\alpha, \beta\}].$$

[Why? As we can use $\{\{2\alpha : \alpha \in A\} : A \in \mathcal{A}\}$, without loss of generality $\bigcup\{A : A \in \mathcal{A}\} = \{2\alpha : \alpha < \lambda\}$ and choose $A_{\alpha, \beta, \gamma} \in [\lambda]^\theta$ for $\alpha < \beta < \gamma < \lambda$ such that $\{\alpha, \beta\} \subseteq A_{\alpha, \beta, \gamma}$ and $\langle A_{\alpha, \beta, \gamma} \setminus \{\alpha, \beta\} : \alpha < \beta < \gamma < \lambda \rangle$ are pairwise disjoint subsets of $\{2\alpha + 1 : \alpha < \lambda\}$, each of cardinality θ and replace \mathcal{A} by $\mathcal{A}^* =: \mathcal{A} \cup \{A_{\alpha, \beta, \gamma} : \alpha < \beta < \gamma < \lambda\}$. Now clause (A), (D), (E) are not affected. Clearly clause $(B)_1$ holds (i.e. $\mathcal{A}^* \subseteq [\lambda]^\theta$ and $A \neq B \in \mathcal{A}^* \Rightarrow |A \cap B| < \sigma$). Also clause (C) is inherited by any extension of the original \mathcal{A} . Lastly for clause $(B)_2$, if $\mathcal{A}' \subseteq \mathcal{A}^*$, $|\mathcal{A}'| < \kappa$, let $\langle A_\zeta : \zeta < \zeta^* \rangle$ be a list of $\mathcal{A}' \cap \mathcal{A}$ as guaranteed by $(B)_2$ and let $\langle A_\zeta : \zeta \in [\zeta^*, \zeta^* + |\mathcal{A}' \setminus \mathcal{A}|] \rangle$ list with no repetitions $\mathcal{A}' \setminus \mathcal{A}$, now check.]

$$\otimes_2 \mathcal{B} \text{ is a basis of } Y^* \text{ of cardinality } \theta^*, \text{ and for part (2), } \mathcal{B} \text{ is as there.}$$

[Why? Straight.]

Let $\mathcal{A} = \{A_\zeta : \zeta < \lambda^*\}$ and $\mathcal{B} = \{b_i : i < \theta^*\}$.

We define a forcing notion P :

$p \in P$ has the form $p = (u, u_*, v, v_*, \bar{w}) = (u^p, u_*^p, v^p, v_*^p, \bar{w}^p)$ such that:

- (α) $u_* \subseteq u \in [\lambda]^{<\kappa}$
- (β) $v_* \subseteq v \in [\lambda^*]^{<\kappa}$
- (γ) $\bar{w} = \bar{w}^p = \langle w_{\zeta, i} : \zeta \in v_* \text{ and } i < \theta^* \rangle = \langle w_{\zeta, i}^p : \zeta \in v_*, i < \theta^* \rangle$
- (δ) $w_{\zeta, i} \subseteq u_*$ and $b_i \cap b_j = \emptyset \Rightarrow w_{\zeta, i} \cap w_{\zeta, j} = \emptyset$; this is toward being Hausdorff
- (ε) $\zeta \in v_* \Rightarrow A_\zeta \subseteq u$
- (ζ) letting $A_\zeta^p = \cup\{w_{\zeta, i} : i < \theta^*\} \cap A_\zeta$ for $\zeta \in v_*^p$ it has cardinality θ and for simplicity even order type θ and for some $\langle \gamma_{\zeta, j}^p : j < \theta \rangle$ list its members with no repetitions we have $w_{\zeta, i}^p \cap A_\zeta^p = \{\gamma_{\zeta, j}^p : j < \theta \text{ and } j \in b_i\}$
- (η) if $\zeta \in v_*^p, i < \theta^*$ and $\xi \in v_*^p$ then the set $\mathcal{U}_{\zeta, \xi, i}^p$ is an open subset (for part (2), clopen subset) of the space Y^* where $\mathcal{U}_{\zeta, \xi, i}^p =: \{j < \theta : \gamma_{\xi, j}^p \in w_{\zeta, i}^p\}$.

$$\oplus \quad \text{convention if } \zeta \in \lambda^* \setminus v_*^p \text{ we stipulate } w_{\zeta, i}^p = \emptyset.$$

The order is: $p \leq q$ iff $u^p \subseteq u^q, u_*^p = u_*^q \cap u^p, v^p \subseteq v^q, v_*^p = v_*^q \cap v^p$ and $\zeta \in v_*^p \Rightarrow w_{\zeta, i}^p = w_{\zeta, i}^q \cap u^p$.

Clearly

(*)₀ P is a partial order.

What is the desired space in V^P ? We define a P -name \dot{X}^* as follows:

set of points $\bigcup \{u_*^p : p \in G_P\}$

The topology is defined by the following basis:

$\{\bigcap_{\ell < n} \mathcal{U}_{\zeta_\ell, i_\ell} : n < \omega, \zeta_\ell < \lambda^*, i_\ell < \theta^*\}$ where

$\mathcal{U}_{\zeta, i}^\ell[G_P] = \bigcup \{w_{\zeta, i}^p : p \in G_P, \zeta \in v_*^p\}$

(for part (2), also their compliments and hence their Boolean combinations)

(*)₁ for $\alpha < \lambda$ and $p \in P$ will have $p \Vdash "\alpha \in \dot{X}^*" \text{ iff } \alpha \in u_*^p \text{ and } p \Vdash "\alpha \notin \dot{X}^*" \text{ iff } \alpha \in u_\alpha^p \setminus u_*^p$

(*)₂ P is κ -complete, in fact if $\langle p_\varepsilon : \varepsilon < \delta \rangle$ is increasing in P and $\delta < \kappa$ then $p = \bigcup_{\varepsilon < \delta} p_\varepsilon$ is an upper bound where $u^p = \bigcup_{\varepsilon < \delta} u^{p_\varepsilon}$, $u_*^p = \bigcup_{\varepsilon < \delta} u_*^{p_\varepsilon}$, $v^p = \bigcup_{\varepsilon < \delta} v^{p_\varepsilon}$, $v_*^p = \bigcup_{\varepsilon < \delta} v_*^{p_\varepsilon}$ and $w_{\zeta, i}^p = \bigcup_{\varepsilon < \delta} w_{\zeta, i}^{p_\varepsilon}$ where $w_{\zeta, i}^{p_\varepsilon} = \bigcup \{w_{\zeta, i}^{p_\varepsilon} : \zeta \in v_*^{p_\varepsilon}, \varepsilon < \delta\}$
[why? straight]

(*)₃ $P' = \{p \in P : \text{if } \zeta < \lambda^* \text{ and } |A_\zeta \cap u^p| \geq \sigma \text{ then } \zeta \in v^p\}$ is a dense subset of P

[why? for any $p \in P$ we define by induction on $\varepsilon \leq \sigma^+$: $p_\varepsilon \in P$, increasingly continuous with ε . Let $p_0 = p$, if p_ε is defined, we define $p_{\varepsilon+1}$ by

$$v^{p_{\varepsilon+1}} = \{\zeta < \lambda^* : \zeta \in v^{p_\varepsilon} \text{ or } |A_\zeta \cap u^{p_\varepsilon}| \geq \sigma\}$$

$$v_*^{p_{\varepsilon+1}} = v_*^{p_\varepsilon}$$

$$u^{p_{\varepsilon+1}} = u^{p_\varepsilon} \cup \bigcup \{A_\zeta : \zeta \in v^{p_{\varepsilon+1}}\}$$

$$u_*^{p_{\varepsilon+1}} = u_*^{p_\varepsilon} (= u_*^p)$$

$$w_{\zeta, i}^{p_{\varepsilon+1}} \text{ is: } w_{\zeta, i}^{p_\varepsilon} \text{ if } \zeta \in v_*^{p_\varepsilon}, i < \theta^*$$

(and there are no other cases).

By assumption (A)(ii), the set $v^{p_{\varepsilon+1}}$ has cardinality $< \kappa$, so $p_{\varepsilon+1}$ belongs to P .

Clearly $p_\varepsilon \leq p_{\varepsilon+1} \in P$.

Now for ε limit let $p_\varepsilon = \bigcup_{\xi < \varepsilon} p_\xi$. So we can carry the definition. Now $p_{\sigma^+} = \bigcup_{\varepsilon < \sigma^+} p_\varepsilon$ is

as required because if $A_\zeta \in \mathcal{A}$, $|A_\zeta \cap u^{p_{\sigma^+}}| \geq \sigma$ then for some $\varepsilon < \sigma^+$, $|A_\zeta \cap u^{p_\varepsilon}| \geq \sigma$ hence $\zeta \in v^{p_{\varepsilon+1}}$ hence $A_\zeta \subseteq u^{p_{\varepsilon+1}} \subseteq u^{p_{\sigma^+}}$.

Note that we use here $\sigma^+ < \kappa$.]

- (*)₄ P satisfies the κ^+ -c.c.
 [Why? Let $p_j \in P$ for $j < \kappa^+$, without loss of generality $p_j \in P'$ for $j < \kappa^+$. Now by the Δ -system lemma for some unbounded $S \subseteq \kappa^+$ and $v^\otimes \in [\lambda^*]^{<\kappa}$, $u^\otimes \in [\lambda]^{<\kappa}$ we have:
 $j \in S \Rightarrow v^\otimes \subseteq v^{p_j}$ & $u^\otimes \subseteq u^{p_j}$ and $\langle v^{p_j} \setminus v^\otimes : j \in S \rangle$ are pairwise disjoint and $\langle u^{p_j} \setminus u^\otimes : j \in S \rangle$ are pairwise disjoint. Without loss of generality $\text{otp}(v^{p_j})$, $\text{otp}(u^{p_j})$ are constant for $j \in S$ and any two p_i, p_j are isomorphic over v^\otimes, u^\otimes (if not clear see 1.5).
 Now for $j_1, j_2 \in S$ the condition p_{j_1}, p_{j_2} are compatible because of the following (*)₅]
- (*)₅ assume $p^1, p^2 \in P$ satisfies
- (i) $u_*^{p^1} \cap (v_*^{p^2} \setminus v_*^{p^1}) = \emptyset$ and $u_*^{p^1} \cap (u_*^{p^2} \setminus u_*^{p^1}) = \emptyset$
 - (ii) $v_*^{p^2} \cap (v_*^{p^1} \setminus v_*^{p^1}) = \emptyset$ and $u_*^{p^2} \cap (u_*^{p^1} \setminus u_*^{p^1}) = \emptyset$
 - (iii) if $\zeta \in v_*^{p^1} \cap v_*^{p^2}$ then $A_\zeta^{p^1} = A_\zeta^{p^2}$ and
 $i < \theta^* \Rightarrow w_{\zeta,i}^{p^1} \cap (u^{p^1} \cap u^{p^2}) = w_{\zeta,i}^{p^2} \cap (u^{p^1} \cap u^{p^2})$
 - (iv)₁ if $\zeta \in v_*^{p^1} \setminus v_*^{p^2}$ then $|A_\zeta \cap u^{p^2}| < \sigma$ or just $|A_\zeta^{p^1} \cap u^{p^2}| < \sigma$
 - (iv)₂ similarly¹ for $\zeta \in v_*^{p^2} \setminus v_*^{p^1}$

then there is $q \in P$ such that:

- (a) $v^q = v^{p^1} \cup v^{p^2}$
- (b) $v_*^q = v_*^{p^1} \cup v_*^{p^2}$
- (c) $u^q = u^{p^1} \cup u^{p^2}$
- (d) $u_*^q = u_*^{p^1} \cup u_*^{p^2}$
- (e) $p^1 \leq q$ and $p^2 \leq q$.

[Why? To define the condition q we just have to define $w_{\zeta,i}^q$ (for $\zeta \in v_*^q = v_*^{p^1} \cup v_*^{p^2}$ and $i < \theta^*$). If $\zeta \in v_*^{p^1} \cap v_*^{p^2}$ we let $w_{\zeta,i}^q = w_{\zeta,i}^{p^1} \cup w_{\zeta,i}^{p^2}$ for $i < \theta^*$.

Now for $\ell = 1, 2$, let $v_*^\ell \setminus v_*^{3-\ell}$ be listed as $\langle \Upsilon(\varepsilon, \ell) : \varepsilon < \varepsilon_\ell^* \rangle$ with no repetitions such that $B_\varepsilon^\ell =: A_{\Upsilon(\varepsilon, \ell)}^{p^\ell} \cap (\bigcup_{\xi < \varepsilon} A_{\Upsilon(\xi, \ell)}^{p^\ell} \cup u^{p^{3-\ell}})$ is of cardinality $< \sigma$.

[Why possible? By the assumption (B)₂ and clause (iv) above.]

Now for each $\zeta \in v_*^{p^{3-\ell}} \setminus v_*^\ell$ we choose by induction on $\varepsilon < \varepsilon_\ell^*$ the sequence $\langle w_{\zeta,i}^{\ell, \varepsilon} : i < \theta^* \rangle$ such that

- 1) $w_{\zeta,i}^{\ell, \varepsilon} \subseteq u^{p^{3-\ell}} \cup \bigcup_{\xi < \varepsilon} A_{\Upsilon(\xi, \ell)}^{p^\ell}$.
- 2) $w_{\zeta,i}^{\ell, \varepsilon}$ is increasingly continuous with ε .
- 3) $w_{\zeta,i}^{\ell, 0} = w_{\zeta,i}^{p^{3-\ell}}$.

¹note that if $p^1, p^2 \in P'$, then clauses (iv)₁, (iv)₂ holds automatically.

$$4) \varepsilon' < \varepsilon \Rightarrow w_{\zeta,i}^{\ell,\varepsilon} \cap (u^{p^{3-\ell}} \cup \bigcup_{\xi < \varepsilon_1} A_{\Upsilon(\xi,\ell)}^{p^\ell}) = w_{\zeta,i}^{\ell,\varepsilon'}.$$

$$5) \text{ if } i < j < \theta^* \text{ and } b_i \cap b_j = \emptyset \text{ (hence } w_{\zeta,i}^{p^\ell} \cap w_{\zeta,j}^{p^\ell} = \emptyset) \text{ then } w_{\zeta,i}^{\ell,\varepsilon} \cap w_{\zeta,j}^{\ell,\varepsilon} = \emptyset.$$

$$6) \{j < \theta : \gamma_{\Upsilon(\varepsilon,\ell),j}^{p^\ell} \in w_{\zeta,i}^{\ell,\varepsilon+1}\} \text{ is an open set in } Y^* \text{ (for part (2): clopen).}$$

For $\varepsilon = 0$ use clause (3) and for limit ε take unions (see clause (2)). Suppose we have defined for ε and let us define for $\varepsilon + 1$. By an assumption above B_ε^ℓ has cardinality $< \sigma$ and so $Z_\varepsilon^\ell = \{j < \theta : \gamma_{\Upsilon(\varepsilon,\ell),j}^{p^\ell} \in B_\varepsilon^\ell\}$ is a subset of θ of cardinality $< \sigma$. Hence, by assumption (E), we can find a sequence $\langle t_j(\varepsilon, \ell) : j \in Z_\varepsilon^\ell \rangle$ such that: $t_j(\varepsilon, \ell) < \theta^*$ and $j \in b_{t_j(\varepsilon,\ell)}$ for $j \in Z_\varepsilon^\ell$ and $\langle b_{t_j(\varepsilon,\ell)} : j \in Z_\varepsilon^\ell \rangle$ is a sequence of pairwise disjoint subsets of Y^* .

Lastly, we let

$$w_{\zeta,i}^{\ell,\varepsilon+1} = w_{\zeta,i}^{\ell,\varepsilon} \cup \{ \gamma_{\Upsilon(\varepsilon,\ell),s}^{p^\ell} : \text{for some } j \in Z_\varepsilon^\ell \text{ we have :} \\ \gamma_{\Upsilon(\varepsilon,\ell),t_j(\varepsilon,\ell)}^{p^\ell} \in w_{\zeta,i}^{\ell,\varepsilon} \text{ and} \\ s \in b_{t_j(\varepsilon,\ell)} \}.$$

Clearly this is O.K. and we are done. Remember that the union of $< \sigma$ set from \mathcal{B} is clopen for part (2).]

(*)₆ in (*)₅ if in addition for $\ell = 1, 2$ we have $Z_\ell \subseteq u^{p^\ell} \setminus u^{p^{3-\ell}}$ such that $(\forall \zeta \in v_*^{p^\ell}) [A_\zeta^{p^\ell} \cap Z_\ell < \sigma]$ then we may add to the conclusion:

$$\ell \in \{1, 2\}, \zeta \in v_*^{p^{3-\ell}} \setminus v_*^{p^\ell}, i < \theta^* \Rightarrow w_{\zeta,i}^q \cap Z_\ell = \emptyset.$$

More generally if $g_\ell : (v_*^{p^{3-\ell}} \setminus v_*^{p^\ell}) \times \theta^* \times Z_\ell \rightarrow \{0, 1\}$ we can add

$$\ell \in \{1, 2\}, \zeta \in v_*^{p^{3-\ell}} \setminus v_*^{p^\ell}, i < \theta^*, \gamma \in Z_\ell \Rightarrow [\gamma \in w_{\zeta,i}^q \leftrightarrow g_\ell(\zeta, i, \gamma) = 1].$$

[Why? During the proof of (*)₅ when for $\zeta \in v_*^{p^{3-\ell}} \setminus v_*^{p^\ell}$, we define $\langle w_{\zeta,i}^{\ell,\varepsilon} : i < \theta^* \rangle$ by induction on ε we add

$$(7) i < \theta^*, \gamma \in Z_\ell \cap (u^{p^{3-\ell}} \cup \bigcup_{\xi < \varepsilon} A_{\Upsilon(\xi,\ell)}^{p^\ell}) \text{ implies } \gamma \in w_{\zeta,i}^{\ell,\varepsilon} \leftrightarrow g_\ell(\zeta, i, \gamma) =$$

1. In the proof when we use clause (E), instead of using $B_\varepsilon^\ell = A_{\zeta(\varepsilon,\ell)}^{p^\ell} \cap (\bigcup_{\xi < \varepsilon} A_{\zeta(\xi,\ell)}^{p^\ell} \cup u^{p^{3-\ell}})$ we use $B_\varepsilon^\ell = A_{\zeta(\varepsilon,\ell)}^{p^\ell} \cap (\bigcup_{\xi < \varepsilon} A_{\zeta(\xi,\ell)}^{p^\ell} \cup u^{p^{3-\ell}} \cup Z_\ell)$ which still has cardinality $< \sigma$.]

Now we come to the main point

(*)₇ in V^P , if $i(*) < \text{cf}(\theta)$ and $X^* = \bigcup_{i < i(*)} X_i$ then some closed $Y \subseteq X^*$ is homeomorphic to Y^* .

[Why? Toward contradiction assume $p^* \in P$ and $p^* \Vdash_P \langle \langle X_i : i < i(*) \rangle \rangle$ is a counterexample to (*)₇].

Without loss of generality $p^* \Vdash_P \langle \langle X_i : i < i(*) \rangle \rangle$ is a partition of X^* , i.e. of $\bigcup \{u_*^p : p \in G_P\}$.

For each $\alpha < \lambda$ let $\langle \langle p_{\alpha,j}, i_{\alpha,j} \rangle : j < \kappa \rangle$ be such that:

- (i) $\langle p_{\alpha,j} : j < \kappa \rangle$ is a maximal antichain of P above p^*
- (ii) $p_{\alpha,j} \Vdash_P \text{“}\alpha \in X_{i_{\alpha,j}}\text{”}$, so $i_{\alpha,j} < i(*)$ and $\alpha \in u_*^{p_{\alpha,j}}$
- (iii) $p^* \leq p_{\alpha,j}$.

Now choose a function F , $\text{Dom}(F) = \lambda$ as follows:

$$F(\alpha) \text{ is } \bigcup \{u^{p_{\alpha,j}} : j < \kappa\}.$$

So we can find $\zeta(*) < \lambda^*$ and $A \subseteq A_{\zeta(*)}$ of order type θ such that: if $\alpha \neq \beta$ are from A then $\alpha \notin F(\beta)$. Let $A = \{\beta_\varepsilon : \varepsilon < \theta\}$ with no repetitions. Now we shall choose by induction on $\varepsilon \leq \theta$, $p_\varepsilon, g_\varepsilon$ and if $\varepsilon < \theta$ also $j_\varepsilon < \kappa$ such that:

- (a) $p_\varepsilon \in P$
 $u^{p_\varepsilon} = u^{p^*} \cup \bigcup_{\varepsilon(1) < \varepsilon} u^{p_{\beta_{\varepsilon(1)}, j_{\varepsilon(1)}}}$
 $u_*^{p_\varepsilon} = u_*^{p^*} \cup \bigcup_{\varepsilon(1) < \varepsilon} u_*^{p_{\beta_{\varepsilon(1)}, j_{\varepsilon(1)}}}$
 $v^{p_\varepsilon} = v^{p^*} \cup \bigcup_{\varepsilon(1) < \varepsilon} v^{p_{\beta_{\varepsilon(1)}, j_{\varepsilon(1)}}}$
 $v_*^{p_\varepsilon} = v_*^{p^*} \cup \bigcup_{\varepsilon(1) < \varepsilon} v_*^{p_{\beta_{\varepsilon(1)}, j_{\varepsilon(1)}}}$
 (so $p_0 = p^*$)
- (b) $j_\varepsilon = \text{Min}\{j < \kappa : p_{\beta_\varepsilon, j} \text{ is compatible with } p_\varepsilon\}$
- (c) g_ε is a function, increasing with ε , from $v_*^{p_\varepsilon} \times \theta^*$ into the family of open subsets of Y^* (for part (2), clopen)
- (d) if $b_{i_1} \cap b_{i_2} = \emptyset$ then $g_\varepsilon(\zeta, i_1) \cap g_\varepsilon(\zeta, i_2) = \emptyset$ (if defined)
- (e) letting $\Upsilon_\varepsilon = \text{otp}\{\xi < \varepsilon : i_{\beta_\varepsilon, j_\varepsilon} = i_{\beta_\xi, j_\xi}\}$ we have for every $\zeta \in v_*^{p_\varepsilon}$ and $i < \theta^*$ and $\xi < \varepsilon$:

$$\beta_\xi \in w_{\zeta, i}^{p_\varepsilon} \Leftrightarrow \Upsilon_\xi \in g_\varepsilon(\zeta, i)$$

- (f) p_ε is increasing continuous.

No problem to carry the definition. As for ε successor, for this $(*)_6$ was prepared. In limit ε take union. In all cases j_ε is well defined by clause (i) above. Let $i^* < i(*)$ be minimal such that the set $Z = \{\varepsilon < \theta : i_{\beta_\varepsilon, j_\varepsilon} = i^*\}$ has cardinality θ . Note: $\zeta(*) \notin v^{p_{\beta_\varepsilon, j_\varepsilon}}$ as $A \cap F(\beta_\varepsilon)$ is a singleton so $|A \cap u^{p_{\beta_\varepsilon, j_\varepsilon}}| \leq 1$ and $p_{\beta_\varepsilon, j_\varepsilon} \in P'$. Now we define p :

$$u^p = u^{p_\theta}$$

$$u_*^p = u_*^{p_\theta}$$

$$v^p = v^{p_\theta} \cup \{\zeta(*)\}$$

$$v_*^p = v_*^{p_\theta} \cup \{\zeta(*)\}$$

$A_{\zeta(*)}^p = \{\beta_\varepsilon : \varepsilon \in Z\}$ and $\gamma_{\zeta(*)}^p$ is the ε -th member of $A_{\zeta(*)}^p$

$w_{\zeta,i}^p$ is

- (α) $w_{\zeta,i}^{p_\theta}$ if $\zeta \in v^{p_\theta}$
- (β) $\{\beta_\varepsilon : \varepsilon \in Z \text{ and } \text{otp}(Z \cap \varepsilon) \in b_i\}$ if $\zeta = \zeta(*)$.

We can easily check that $p \in P$ and $p^* \leq p_{\beta_\varepsilon, j_\varepsilon} \leq p \in P$ (but we do not ask $p_\varepsilon \leq p$). Clearly p forces that $\{\beta_\varepsilon : \varepsilon \in Z\}$ is included in one X_i .

Let $g : \theta \rightarrow \lambda$ be $g(\xi) = \beta_\varepsilon$ when $\xi < \theta, \varepsilon \in Z, \text{otp}(Z \cap \varepsilon) = \xi$. Now $p \geq p^*$ and we are done by $(*)_8$ below.]

- ($*$)₈ if $p \in P$ and $\zeta \in v_*^p$ then
 - $p \Vdash$ “the mapping $j \mapsto \gamma_{\zeta,j}^p$ for $j < \theta$ is a homeomorphism from Y^* onto the closed subspace $\underline{X} \restriction \{\gamma_{\zeta,j}^p : j < \theta\}$ of \underline{X} ”
 - [Why? Let $p \in G, G \subseteq P$ is generic over V .
 - (α) If $b \in \mathcal{B}$, then for some open set \mathcal{U} of \underline{X} (clopen for part (2)) we have

$$\mathcal{U} \cap \{\gamma_{\zeta,j}^p : j < \theta\} = \{\gamma_{\zeta,j}^p : j \in b\}$$

[Why? As $b = b_i$ for some $i < i(*)$ and p forces that $w_{\zeta,i} \cap \{\gamma_{\zeta,j}^p : j < \theta\} = \{\gamma_{\zeta,j}^p : j \in b_i\}$.]

- (β) If b is an open set for Y^* , then for some open subset \mathcal{U} of \underline{X} we have

$$\mathcal{U} \cap \{\gamma_{\zeta,j}^p : j < \theta\} = \{\gamma_{\zeta,j}^p : j \in b\}$$

[Why? As $b = \bigcup_{i \in Z} b_i$ for some $Z \subseteq \theta^*$ and apply clause (α)]

- (γ) if \mathcal{U} is an open subset of \underline{X} and $\gamma_{\zeta,j(*)}^p \in \mathcal{U}$ (so $\zeta \in u_*^p$), then for some $i(*) < \theta^*$ we have

$$\gamma_{\zeta,j(*)}^p \in w_{\zeta,i(*)}^p \cap \{\gamma_{\zeta,j}^p : j < \theta\} \subseteq \mathcal{U}_{\zeta,i(*)} \cap \{\gamma_{\zeta,j}^p : j < \theta\} \subseteq \mathcal{U}.$$

[Why? By the definition of the topology \underline{X} we can find $n < \omega, \xi_\ell < \lambda^*$

and $i_\ell < \theta^*$ for $\ell < n$ such that $\gamma_{\zeta,j(*)}^p \in \bigcap_{\ell < n} \mathcal{U}_{\xi_\ell, i_\ell}[G] \subseteq \mathcal{U}$. We

can find $q \in P$ such that $p \leq q$ and $\xi_\ell \in v_*^q$ for $\ell < n$. For each $\ell < n$, by clause (η) in the definition of P we have $\mathcal{U}_{\zeta, \xi_\ell, j_\ell}^q$ is an open set for Y^* , and necessarily $j(*) \in \mathcal{U}_{\zeta, \xi_\ell, j_\ell}^q$. Let $i(*)$ be such that

$j(*) \in b_{i(*)} \subseteq \bigcap_{\ell < n} \mathcal{U}_{\zeta, \xi_\ell, j_\ell}^q$ hence $\gamma_{\zeta,j(*)}^p \in \mathcal{U}_{\zeta, i(*)}[G] \cap \{\gamma_{\zeta,j}^p : j < \theta\} \subseteq$

$\bigcap_{\ell < n} \mathcal{U}_{\xi_\ell, j_\ell}[G] \subseteq \mathcal{U}$ as required. So $i(*)$ is as required.]

- (δ) $\{\gamma_{\zeta,j}^p : j < \theta\}$ is a closed subset of \underline{X}
- [Why? Let $\beta \in \lambda \setminus \{\gamma_{\zeta,j}^p : j < \theta\}$ and let $p \leq q \in P$; it suffices to find

$q^+, q \leq q^+ \in P$ and $\xi \in v_*^{q^+}$ and $i < \theta^*$ such that $\beta \in u^{q^+} \setminus u_*^{q^+}$ or $\beta \in w_{\xi,i}^{q^+}$ and $w_{\xi,i}^{q^+} \cap \{\gamma_{\zeta,j}^p : j < \theta\} = \emptyset$. If $\beta \notin u_*^q$ define q^+ like q except that $u^{q^+} = u^q \cup \{\beta\}$ (but $u_*^{q^+} = u_*^q$). So without loss of generality $\beta \in u_*^q$.

We can find a set $u \subseteq u_*^q$ such that $\beta \in u$, $A_\zeta^q \cap u = \emptyset$ and $\zeta' \in v_*^q \Rightarrow \{j < \theta : \gamma_{\zeta',j}^q \in u\}$ is an open subset of Y , (just as in the proof of $(*)_5$; for part (2) we ask “clopen subset of Y ”). By \otimes_1 we can find $\xi \in \lambda^* \setminus v^q$ such that $\{\emptyset\} = A_\xi \cap u_*^q$ (why? apply \otimes_1 with $\alpha < \beta \in \lambda \setminus u^q$ and $B = u^q$) and let $\gamma_{\varepsilon,i} \in A_\xi$ for $i < \theta$ be increasing. We define q^+ as follows.

$$v^{q^+} = v^q \cup \{\xi\}$$

$$v_*^{q^+} = v_*^q \cup \{\xi\}$$

$$u^{q^+} = u^q \cup A_\xi$$

$$u_*^{q^+} = u_*^q \cup \{\gamma_{\xi,j} : j < \theta\}$$

$w_{\zeta,i}^{q^+}$ is $w_{\zeta,i}^q$ if $\zeta \in v_*^q$ and is $\{\gamma_{\xi,j} : j \in b_i\} \cup u$ if $\zeta = \xi$ & $0 \in b_i$ and is $\{\gamma_{\xi,j} : j \in b_i\}$ if $\zeta = \xi$ & $0 \notin b_i$.]

Lastly, we would like to know that \underline{X} is a Hausdorff space. We prove more

(*)₉ In V^P if $u_1 \subseteq u_2 \in [\lambda]^{<\sigma}$ then for some ζ, i we have

$$w_{\zeta,i} \cap u_2 \cap \underline{X} = u_1 \cap \underline{X}$$

[Why? Let $p_0 \in P$ force that $u_1 \subseteq u_2$ form a counterexample, as P is κ -complete some $p_1 \geq p_0$ forces $u_1 = u_2$, $u_2 = u_2$ and $p_1 \in P'$. Necessarily $u_2 \subseteq u_*^{p_1}$, as in the proof of $(*)_8(\delta)$.

Let $\zeta(*) \in \lambda^* \setminus v^{p_1}$ be such that $A_{\zeta(*)} \cap u^{p_1} = \emptyset$ (as in the proof of $(*)_8(\delta)$). Let $\gamma_{\zeta(*),j} \in A_{\zeta(*)}$, for $j < \theta$ be increasing. Let $u \subseteq u_*^{p_1}$ be such that $u \cap u_2 = u_1$ and $\zeta' \in v_*^{p_1} \Rightarrow \{j < \theta : \gamma_{\zeta',j}^{p_1} \in u\}$ is clopen in Y (exists as in the proof of $(*)_5$) and define $q \in P$:

$$u^q = u^{p_1} \cup u_2$$

$$u_*^q = u_*^{p_1} \cup (u_2 \setminus u^{p_1})$$

$$v^q = v^{p_1} \cup \{\zeta(*)\}$$

$$v_*^q = v_*^{p_1} \cup \{\zeta(*)\}$$

$w_{\zeta,i}^q$ is: $w_{\zeta,i}^{p_1}$ if $\zeta \in v^q$, is $\{\gamma_{\zeta(*),j} : j \in b_i\} \cup u$ if $\zeta = \zeta(*)$ & $0 \in b_i$ and is $\{\gamma_{\zeta(*),j} : j \in b_i\}$ if $\zeta = \zeta(*)$ & $0 \notin b_i$.

Together all is done. $\square_{1.2}$

* * *

Now when are the assumptions of 1.2 hold?

1.4 Claim. *Assume*

- (a) $\mathfrak{a} \in [Reg \cap \mu \setminus \kappa]^\theta, J = [\mathfrak{a}]^{<\sigma}$
- (b) $\Pi\mathfrak{a}/J$ is $(\lambda^*)^+$ -directed,
- (c) $\lambda \geq \mu$ is singular,
- (d) $\lambda^* > \lambda > \kappa^{<\kappa} = \kappa > \theta$;
- (e) $\lambda^* < 2^\lambda$ is regular.

Then

- (f) In $V_1 = V^{\text{Levy}(\lambda^*, 2^\lambda)}$ we have (a),(c),(d) and (e) and $2^\lambda = \lambda^*$ and
- (g) the assumptions (A)(i), (B_1) , (B_2) , (C) of Theorem 1.2 hold (recall (A)(i) means we omit θ^* and $(\forall \alpha < \kappa)(|\alpha|^\sigma < \kappa)$).

Proof. Let $\mathfrak{a} = \{\lambda_\varepsilon : \varepsilon < \theta\}$ without repetitions; without loss of generality $\lambda_i > \kappa^{++}$. Let $J' = [\theta]^{<\sigma}$. By [Sh:g, Ch.II,1.4] (at least the proof, see below) in V we can find $\langle f_\alpha : \alpha < \lambda^* \rangle$ such that (more but irrelevant here)

- (*)₀ Assume J' is an ideal on \mathfrak{a} , $\Pi\mathfrak{a}/J'$ is $(\lambda^*)^+$ -directed, $\lambda^* > \sup(\mathfrak{a})$. Then we can find $\langle f_\alpha : \alpha < \lambda^* \rangle$ such that $f_\alpha \in \prod_{\varepsilon < \theta} \lambda_\varepsilon$ and for every $Z \in [\lambda^*]^{<\kappa}$ for

some sequence $\bar{a} = \langle a_\alpha : \alpha \in Z \rangle$ such that $a_\alpha \in J'$ for $\alpha \in Z$ and some well ordering $<^*$ of Z we have

- (i) $\alpha_1 \in Z$ & $\alpha_2 \in Z$ & $\varepsilon_1 < \theta$ & $\varepsilon_2 < \theta$ & $f_{\alpha_1}(\varepsilon_1) = f_{\alpha_2}(\varepsilon_2) \rightarrow \varepsilon_1 = \varepsilon_2$
- (ii) $\alpha \in Z$ & $\beta \in Z$ & $\alpha <^* \beta$ & $\varepsilon \in \theta \setminus a_\beta \Rightarrow f_\alpha(\varepsilon) \neq f_\beta(\varepsilon)$.

[Why? There we get only: for some $\langle f_\alpha : \alpha < \lambda^* \rangle$ as above with (i) + (ii) replaced by: for every $Z \in [\lambda^*]^{<\kappa}$ (even $Z \in [\lambda]^\mu$ if $\mathfrak{a} \subseteq \mu'$ has order type $\leq \sigma$) we can find $\bar{a} = \langle a_\alpha : \alpha \in Z \rangle$ such that $a_\alpha \in J'$ and $\alpha \neq \beta$ & $\alpha \in Z$ & $\beta \in Z$ & $\varepsilon \in \theta \setminus a_\alpha \setminus a_\beta \Rightarrow f_\alpha(\varepsilon) \neq f_\beta(\varepsilon)$. Clause (i) is easy, just replace f_α by f'_α which is defined by $f'_\alpha(\lambda_\varepsilon) = \theta \times f_\alpha(\lambda_\varepsilon) + \varepsilon$. We shall prove $\langle f_\alpha : \alpha < \lambda^* \rangle$ is as required. So let $Z \in [\lambda^*]^{<\kappa}$. We can choose by induction on $\zeta, Z_\zeta \subseteq Z$ increasingly continuous in ζ such that $Z_0 = \emptyset, |Z_{\zeta+1} \setminus Z_\zeta| \leq \sigma, [Z_\zeta \neq Z \Rightarrow Z_\zeta \neq Z_{\zeta+1}]$ and $\alpha \in Z_{\zeta+1} \setminus Z_\zeta$ & $\varepsilon \in a_\alpha$ & $\beta \in Z$ & $\varepsilon \notin a_\beta$ & $f_\alpha(\varepsilon) = f_\beta(\varepsilon) \rightarrow \beta \in Z_{\zeta+1}$. As $|a_\alpha| < \sigma$ and $(\forall \varepsilon < \theta)(\forall \gamma)(\exists^{\leq 1} \beta)(\varepsilon \notin a_\beta \text{ & } f_\beta(\varepsilon) = \gamma)$ there is no problem, (in fact if σ is regular we can ask $< \sigma$). Now list Z as $\langle \alpha_\xi : \xi < \xi^* \rangle$ such that $\{\xi : \alpha_\xi \in Z_{\zeta+1} \setminus Z_\zeta\}$ is a convex set of order type of its cardinality so $\leq \sigma$, which is above $\{\xi : \alpha_\xi \in Z_\zeta\}$ and $\alpha_\xi \in Z_{\zeta+1} \setminus Z_\zeta \Rightarrow \sigma > |\cup \{a_\varepsilon : \varepsilon \in Z_{\zeta+1} \setminus Z_\zeta\}|$

and $\alpha_\varepsilon \leq \alpha_\xi\}$. Define a well ordering $<^*$ of Z by $\alpha_\varepsilon <^* \alpha_\xi \equiv \xi < \varepsilon$. Now we define $a'_\alpha \in J'$ for $\alpha \in Z$ as follows: if $\alpha = \alpha_\xi \in Z_{\zeta+1} \setminus Z_\zeta$ then $a'_\alpha = \cup\{a_{\alpha_\varepsilon} : \alpha_\varepsilon \in Z_{\zeta+1} \setminus Z_\zeta \text{ and } \varepsilon \leq \xi\}$. Now suppose for $\langle a'_\alpha : \alpha \in Z \rangle$ fails clause (ii), so there are $\varepsilon < \theta$ and $\alpha < \beta$ from Z which exemplifies this. As $a_\alpha \subseteq a'_\alpha$ and the choice of $\langle a_\alpha : \alpha \in Z \rangle$ necessarily $\varepsilon \in a_\alpha$ and we get easy contradiction.]

Clearly in V_1 we have (a),(c),(d) and $(*)_0$ above (and we can forget V and (b), recall (b) says “ Πa is $(\lambda^*)^+$ -directed”, on the existence of \bar{f} as in $(*)_0$, see [Sh:g, Ch.VIII,§5]).

We can in V_1 list $\langle h_\alpha : \alpha < \lambda^* \rangle$ the functions $h : \lambda \rightarrow [\lambda]^\kappa$. Now for each $\zeta < \lambda^*$ we define a function $g_\zeta : \kappa^{++} \rightarrow [\kappa^{++}]^{\leq \kappa}$ by

$$g_\zeta(\gamma) = \left\{ \beta < \kappa^{++} : \text{for some } \varepsilon_1, \varepsilon_2 < \theta \text{ we have} \right. \\ \left. f_\alpha(\varepsilon_1) \times \kappa^{++} \times \theta + \beta \times \theta + \varepsilon_1 \in \right. \\ \left. h_\zeta[f_\zeta(\varepsilon_2) \times \kappa^{++} \times \theta + \gamma \times \theta + \varepsilon_2] \right\}.$$

So we can for each $\zeta < \lambda^*$ find $Z_\zeta \in [\kappa^{++}]^{\kappa^{++}}$ such that

$$\beta_1 \neq \beta_2 \in Z_\zeta \Rightarrow \beta_1 \notin g_\zeta(\beta_2).$$

For $\zeta < \lambda^*$ let $A_\zeta = \{f_\zeta(\varepsilon) \times \kappa^{++} \times \theta + \beta \times \theta + \varepsilon : \varepsilon < \theta \text{ and } \beta < \kappa^{++} \text{ is the } \varepsilon\text{-th member of } Z_\zeta\}$.

Now we shall check.

Let $\mathcal{A} = \{A_\zeta : \zeta < \lambda^*\}$. Clearly

$$(*)_1 \quad A_\zeta \in [\lambda]^\theta \text{ (hence } \mathcal{A} \subseteq [\lambda]^\theta)$$

$$(*)_2 \quad \zeta_1 \neq \zeta_2 \Rightarrow |A_{\zeta_1} \cap A_{\zeta_2}| < \sigma$$

[Why? Let $\alpha \in A_{\zeta_1} \cap A_{\zeta_2}$ so for some $\ell = 1, 2$ we have $\alpha = f_{\zeta_\ell}(\varepsilon_\ell) \times \kappa^{++} \times \theta + \beta_\ell \times \theta + \varepsilon_\ell$ with $\beta_\ell < \kappa^{++}, \varepsilon_\ell < \theta$. Clearly this implies $\varepsilon_0 = \varepsilon_1, \beta_1 = \beta_2, f_{\zeta_1}(\varepsilon_\ell) = f_{\zeta_2}(\varepsilon_\ell)$, and $\text{otp}(\beta_\ell \cap Z_{\zeta_\ell}) = \varepsilon_\ell$, so β_ℓ depends just on ζ_ℓ and ε_ℓ (not on α) and by (i) of $(*)_0$ also ε_ℓ is determined by $\zeta_\ell, f_{\zeta_\ell}(\varepsilon_\ell)$ hence $|A_{\zeta_1} \cap A_{\zeta_2}| < |\{\varepsilon < \theta : f_{\zeta_1}(\varepsilon) = f_{\zeta_2}(\varepsilon)\}| < \sigma$ as $\zeta_1 < \zeta_2 \rightarrow f_{\zeta_1} <_J f_{\zeta_2}$ recalling $J = [\theta]^{<\sigma}$.]

$$(*)_3 \quad |\mathcal{A}| = \lambda^*$$

[Why? By the choice of \mathcal{A} and $(*)_1 + (*)_2$.]

$$(*)_4 \quad \text{if } F : \lambda \rightarrow [\lambda]^{\leq \kappa}, \text{ then some } A \in \mathcal{A} \text{ is } F\text{-free}$$

[Why? For some α we have $F = h_\alpha$, so Z_α, A_α were chosen to make this true.]

$$(*)_5 \quad \text{if } \mathcal{A}' \in [\mathcal{A}]^{<\kappa}, \text{ then we can list } \mathcal{A}' \text{ as } \{A_{\zeta_i} : i < i(*)\} \text{ such that}$$

$$|A_{\zeta_i} \cap \bigcup_{j < i} A_{\zeta_j}| < \sigma$$

[Why? Let $\mathcal{A}' = \{A_\zeta : \zeta \in Z\}$ where $Z \subseteq \lambda^*, |Z| < \kappa$, so by $(*)_0$ we can

find $\langle a_\alpha : \alpha \in Z \rangle, <^*$ as there. Let $Z = \{\zeta_i : i < i(*)\}$ be $<^*$ -increasing with i and so

$$A_{\zeta_i} \cap \bigcup_{j < i} A_{\zeta_j} \subseteq \{f_{\zeta_i}(\varepsilon) \times \kappa^{++} \times \theta + \theta \times \beta_{\zeta_i, \varepsilon} + \varepsilon : \varepsilon \in a_{\zeta_i}\}$$

which has cardinality $< \sigma$ where $\beta_{\zeta_i, \varepsilon}$ is the ε -th member of Z_α .]

So clause (A)(i) holds by our assumption (note, θ^* does not appear here), clause $(B)_1$ holds by $(*)_1 + (*)_2$ and $(B)_2$ holds by $(*)_5$ and lastly (C) holds by $(*)_4$. $\square_{1.4}$

1.5 Claim. *We can weaken the assumption 1.2 omitting in part (2) the “closed under union of $< \sigma$ ” and by omitting (A)(ii) and by replacing (E) by $(E)^-$, i.e. having:*

- (A)(i) $\lambda > \kappa > \theta \geq \sigma \geq \aleph_0$ and $\kappa = \kappa^{<\kappa}$
(i.e. this is (A)(i) without (A)(ii) i.e. omitting “ $(\forall \alpha < \kappa)(|\alpha|^\sigma < \kappa), \kappa > \theta^ \geq \theta$ ”)*
- $(E)^-$ *if Y_0, Y_1 are disjoint subsets of Y^* each with $< \sigma$ points, then there are open disjoint sets $\mathcal{U}_0, \mathcal{U}_1$ of Y^* such that $Y_0 \subseteq \mathcal{U}_0, Y_\theta \subseteq Y_1$.*

Proof. We indicate the changes.

We can further demand from $\langle b_i : i < \theta^* \rangle$ that

$$\boxtimes_1 \quad b_{2i} \cap b_{2i+1} = \emptyset \text{ and if } b_{i_0} \cap b_{i_1} = \emptyset \text{ then for some } j \text{ we have } (b_{2j}, b_{2j+1}) = (b_{i_0}, b_{i_1}).$$

In the definition of P we replace clause (δ) by

$$(\delta)^- \quad w_{\zeta, i} \subseteq u_* \text{ and } w_{\zeta, 2i} \cap w_{\zeta, 2i+1} = \emptyset.$$

However, as we have weakened assumption (A), the κ^+ -c.c. may fail. So we define: we say (f, g) is an isomorphism from $p \in P$ onto $q \in P$ if:

- (i) f is a one-to-one mapping from u^p onto u^q
- (ii) g is a one-to-one mapping from v^p onto v^q
- (iii) f maps u_*^p onto u_*^q
- (iv) g maps v_*^p onto v_*^q
- (v) if $\zeta \in v_*^p$ then $A_{g(\zeta)} = \{f(\beta) : \beta \in A_\zeta\}$
- (vi) if $\zeta \in v_*^p$ and $j < \theta$ then $\gamma_{g(\zeta), j}^q = f(\gamma_{\zeta, j}^p)$
- (vii) if $\zeta \in v_*^p$ and $i < \theta^*$ then

$$w_{g(\zeta), i}^q = \{f(\beta) : \beta \in w_{\zeta, i}^p\}.$$

We say p, q are isomorphic if such (f, g) exists. Clearly being isomorphic is an equivalent relation. Let χ be large enough and \mathfrak{C} be an elementary submodel of $(\mathcal{H}(\chi), \in, <^*)$ of cardinality κ such that $\lambda, \kappa, \theta^*, \theta, \sigma, Y^*, \langle b_i : i < \theta^* \rangle, \mathcal{A}, P$ belong to \mathfrak{C} and ${}^{\kappa}>\mathfrak{C} \subseteq \mathfrak{C}$. Let

$$Q = \{p \in P : \text{for some } q \in P \cap \mathfrak{C} \text{ we have } p, q \text{ are isomorphic}\}.$$

In the rest of the proof P is replaced by Q , each time we construct a condition we have to check if it belongs to Q .

The only place we use $(\forall \alpha < \kappa)(|\alpha|^\sigma < \kappa)$ is in the proof of $(*)_3$. So omit $(*)_3$, and this requires us just to improve the proof of $(*)_4$. Let $p_j \in Q$ for $j < \kappa^+$ and let $v_j = \{\zeta < \lambda^* : A_\zeta \cap \mathcal{U}^{p_j} \text{ has cardinality } \geq \sigma\} \cup v^{p_j}$, so clearly $|v_j| \leq \kappa$ and $v^{p_j} \subseteq v_j$.

For some stationary $S \subseteq \{\delta < \kappa^+ : \text{cf}(\delta) = \kappa\}$, the conditions p_j for $j \in S$ are pairwise isomorphic and $j \in S$ implies $v^{p_j} \cap (\bigcup_{i < j} v_i) = v^\otimes$ and $u^{p_j} \cap (\bigcup_{i < j} (u^{p_i} \cup \bigcup_{\zeta \in v_i} A_\zeta)) = u^\otimes$. Also without loss of generality for $j_1, j_2 \in S$ the isomorphism (f, g) from p_{j_1} to p_{j_2} satisfies $f \upharpoonright u^\otimes = \text{id}_{u^\otimes}, g \upharpoonright v^\otimes = \text{id}_{v^\otimes}$. Now for $i, j \in S, p_i, p_j$ are compatible by $(*)_5$ in the proof of 1.2.

In the proof of $(*)_5$ and $(*)_6$ (hence $(*)_7$), clause $(E)^-$ gives us less but the change in the definition of P (weakening (δ) to $(\delta)^-$) demands less and they fit.

Lastly, for proving “ \bar{X} is Hausdorff”, clause $(\delta)^-$ is weaker but as Y^* is Hausdorff (and the choice of $\langle b_i : i < \theta^* \rangle$) there is no problem. $\square_{1.5}$

1.6 Comment 1) We could make in 1.2 only some of the changes from 1.5, e.g. allow $(A)^-$ and (E) .

2) In 1.2(1) can we make the space regular (T_3) ?

In view of 1.2(2) this may be not so interesting, still let $R_0 \subseteq \{(i, j) : b_i \cap b_j = \emptyset\}$ (so to include generalizing in 1.5 we choose $R_0 \subseteq \{(2i, 2i + j) : i < \theta^*\}$) and $R_1 \subseteq \{(i, j) : b_i \cup b_j = Y^*\}, R_2 \subseteq \{(i, j) : b_i \subseteq b_j\}$.

We need: for $i_0 < \theta^*, j < \theta$ such that $j \in b_{i_0}$ there are $i_1, i_2 < \theta^*$ such that $j \in b_{i_1}, b_{i_1} \subseteq b_{i_0}, b_{i_0} \cup b_{i_2} = Y^*, b_{i_1} \cap b_{i_2} = \emptyset$ and moreover $(i_0, i_1) \in R_2, (i_0, i_2) \in R_1, (i_1, i_2) \in R_0$.

Then we should change the definition of P , clause (δ) to

- $(\delta)^- (a) \quad w_{\zeta, i} \subseteq u_*$
- $(b) \quad (i, j) \in R_0 \Rightarrow w_{\zeta, i} \cap w_{\zeta, j} = \emptyset;$
- $(c) \quad (i, j) \in R_1 \Rightarrow w_{\zeta, i} \cup w_{\zeta, j} = u_*$
- $(d) \quad (i, j) \in R_2 \Rightarrow w_{\zeta, i} \subseteq w_{\zeta, j}.$

As \bar{b} can be with repetition without loss of generality $\langle R_0, R_1, R_2 \rangle$ have a tree structure. That is without loss of generality there is a partial function $g^* : \theta^* \rightarrow \theta^*$ such that $g^*(2i) = g^*(2i + 1) < i$ (so $2i \in \text{Dom}(g^*) \leftrightarrow 2i + 1 \in \text{Dom}(g^*)$) and $R_0 = \{(2i, 2i + 1) : 2i \in \text{Dom}(g^*) \text{ and } i = 0 \bmod 3\}, R_1 = \{(g^*(2i), 2i + 1) : 2i \in \text{Dom}(g^*) \text{ and } i = 1 \bmod 3\}, R_2 = \{(g^*(2i), 2i) : 2i \in \text{Dom}(g^*) \text{ and } i = 2 \bmod 3\}.$

So we need to have the following property of Y^* (this will be used in the proof of $(*)_5, (*)_6$ hence $(*)_7$ dealing with $w_{\zeta, i}^{\varepsilon, \ell}$ by induction on $i < \theta^*$)

\boxtimes_{Y^*} if $k \in \{0, 1, 2\}$, \mathcal{U}_k is open for Y^* , $Z \subseteq Y$, $|Z| < \sigma$ and $Z \cap \mathcal{U} = Z \cap (\text{closure } (\mathcal{U}))$ and $Z'_\ell \subseteq Z$ for $\ell = 0, 1, 2$ satisfy $Z'_1 \subseteq Z'_0$, $Z'_0 \cup Z'_2 = Z$, $Z'_1 \cap Z'_2 = \emptyset$ and $Z'_k = \mathcal{U}_k \cap Z$, then we can find open subsets \mathcal{U}_ℓ of Y^* for $\ell \in \{0, 1, 2\} \setminus \{k\}$ such that $\mathcal{U}_0 \subseteq \mathcal{U}_1$, $\mathcal{U}_0 \cup \mathcal{U}_2 = Y^*$ and $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$ and $Z'_\ell = \mathcal{U}_\ell \cap Z$ for $\ell = 1, 2$.

If $\sigma = \aleph_0$ this requirement on Y^* follows from Y^* being Hausdorff. Also clause (E) of 1.2 implies \boxtimes_{Y^*} .

3) In 1.4, as indicated in the proof, we can replace in the assumption (b) + (e), i.e. “ $\Pi\mathfrak{a}/J$ is $(\lambda^*)^+$ -directed, $\lambda^* < 2^\lambda$ ” is regular by:

(*) there is $\bar{f} = \langle f_\alpha : \alpha < \lambda^* \rangle$, $f_\alpha \in \pi\mathfrak{a}$ such that for every $Z \in [\lambda^*]^{<\kappa}$ we can find $\langle a_\alpha : \alpha \in Z \rangle$, $a_\alpha \in J$ such that $\alpha \neq \beta \in Z$ & $\varepsilon \in \theta \setminus a_\alpha \setminus a_\beta \Rightarrow f_\alpha(\varepsilon) \neq f_\beta(\varepsilon)$.

4) By the proof of 1.4, if $(\mathfrak{a}, J = [\mathfrak{a}]^{<\sigma})$ and \bar{f} are as in (*) of 1.6(3)) then

(*)' there is $\bar{f}' = \langle f'_\alpha : \alpha < \lambda^* \rangle$, $f'_\alpha \in \Pi\mathfrak{a}$ such that for every $Z \in [\lambda^*]^{<\kappa}$ we can find $\langle a_\alpha : \alpha \in Z \rangle$, $a_\alpha \in J$ and well ordering $<^*$ of Z such that $\alpha <^* \beta \in Z$ & $\varepsilon \in \theta \setminus a_\beta \Rightarrow f'_\alpha(\varepsilon) \neq f'_\beta(\varepsilon)$ (in fact $\bar{f}' = \bar{f}$).

5) See more 4.13: for more colours.

§2 CONSISTENCY FROM SUPERCOMPACT

In the first section we got consistency results concerning Arhangelskii's problem but using pcf statement of unclear status (they come from 1.4); this is very helpful toward finding the consistency strength, and unavoidable if e.g. we like CH to fail (see §3), but it does not give a well grounded consistency result. Here relying on Theorem 1.2 of the first section, we get consistency results using supercompact cardinals. First we give a sufficient condition for clause (C) of Theorem 1.2 which is reasonable under instances of G.C.H. We then (2.2) quote a theorem of Hajnal Juhasz Shelah [HJSh 249], [Sh:F267] (for $\sigma = \aleph_0, \sigma > \aleph_0$, respectively) and from it (in claim 2.3), in the natural cases, prove that the assumptions of 1.2 hold deducing (in 2.4) the consistency of CH + there is a T_3 -space X with clopen basis with $\aleph_{\omega+1}$ point such that $X \rightarrow (\text{Cantor set})_{\aleph_0}^1$ starting with a supercompact cardinal. This gives a (consistent) negative answer to Arhangelskii's problem. We can even make it compact.

2.1 Observation: If clauses² (A)(i) + (B)₁ of Theorem 1.2 holds, then clause (C) there follows from

(C)⁺ if $\langle Y_i : i < \kappa^+ \rangle$ is a partition of λ then for some $A \in \mathcal{A}$ and $i < \kappa^+$ we have $A \subseteq Y_i$.

Proof. Let $F : \lambda \rightarrow [\lambda]^{\leq \kappa}$ be given. Choose by induction on $\zeta \leq \lambda$ a set $U_\zeta \subseteq \lambda$ and $g_\zeta : U_\zeta \rightarrow \kappa^+$, both increasingly continuous with ζ such that:

- (*) (i) if $\alpha \in U_\zeta$ then $F(\alpha) \subseteq U_\zeta$ and
- (ii) if $\alpha \in U_\zeta$ then $F(\alpha) \setminus \{\alpha\} \subseteq \{\beta \in U_\zeta : g_\zeta(\beta) \neq g_\zeta(\alpha)\}$.

For $\zeta = 0$ let $U_\zeta = \emptyset = g_\zeta$, for ζ limit take unions. If $U_\zeta = \lambda, U_{\zeta+1} = U_\zeta, g_{\zeta+1} = g_\zeta$, otherwise let $\alpha_\zeta = \min\{\lambda \setminus U_\zeta\}$ and let $W_\zeta \in [\lambda]^{\leq \kappa}$ be such that $\alpha_\zeta \in W_\zeta$ and $(\forall \alpha \in W_\zeta)[F(\alpha) \subseteq W_\zeta]$. Let $\varepsilon_\zeta = \sup\{g_\zeta(\beta) : \beta \in U_\zeta \cap W_\zeta\}$ so $\varepsilon_\zeta < \kappa^+$ and let $U_{\zeta+1} = U_\zeta \cup W_\zeta, g_{\zeta+1}$ extends g_ζ such that $g_{\zeta+1} \upharpoonright (W_\zeta \setminus U_\zeta)$ is one to one with range $[\varepsilon, \varepsilon + \kappa)$.

Now applying (C)⁺ to the partition which $\bigcup_{\zeta} g_\zeta$ defines, we get some $A \in \mathcal{A}$ on

which $\bigcup_{\zeta} g_\zeta$ is constant so by (ii) we are done. □_{2.1}

By [HJSh 249], [Sh:F276]

2.2 Claim. Assume $V \models GCH$ (for simplicity) and $\sigma < \chi < \chi_0^{<\chi} \leq \kappa < \mu < \mu^+ = \lambda$ and $\sigma, \chi, \chi_0, \kappa, \lambda$ are regular, $\text{cf}(\mu) = \sigma, \chi$ is a supercompact cardinal $> \sigma$ (or just λ -supercompact), e.g. $\mu = \chi_0^{+\sigma}$.

Then for some forcing notion, σ -complete of cardinality χ_0 , in $V^P, 2^\sigma = \sigma^+ = \theta, 2^{\sigma^+} = \chi_0 = \sigma^{++}$ (and GCH holds) and some $\langle B_\delta : \delta \in S \rangle$ satisfies:

²(actually from (B)₁, only “ $(B)_1^- \mathcal{A} \subseteq [\lambda]^\theta$ ” is used; as we do not change \mathcal{A} and the cardinals this is O.K.

- (*) $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \sigma^+\}$ is stationary,
 $B_\delta \subseteq \delta$, $\text{otp}(B_\delta) = \theta^+$ and $\delta_1 \neq \delta_2 \in S \Rightarrow |B_{\delta_1} \cap B_{\delta_2}| < \sigma$ and
 $\theta^* < \kappa = \kappa^{<\kappa} < \lambda$, (hence \Diamond_S , so if $\mu = \chi_0^{+\omega}$ then $\lambda = \aleph_{\omega+1}$).

What we need is getting in such model, condition $(C)^+$ of 2.1 which also is from [Sh:F276] but for completeness we shall prove it.

2.3 Claim. Assume

- (a) $\langle B_\delta : \delta \in S \rangle, \sigma, \kappa, \mu, \lambda$ are as in the conclusion (*) of the previous claim and
 (b) S reflects in no ordinal of cofinality $\leq \kappa$ (holds automatically if $\kappa < \sigma^{+\sigma}$, see [Sh 108], [Sh 88a]), but see 2.5, 2.6.

Then without loss of generality $\sigma, \theta =: \sigma^+, \lambda, \mathcal{A} = \{B_\delta : \delta \in S\}$ satisfies the requirements in Theorem 1.2 (in V^P).

Proof. Without loss of generality “ $\delta \in S \Rightarrow \mu^\omega$ divides δ ”, and as we are assuming GCH and $\delta \in S \Rightarrow \text{cf}(\delta) = \sigma^+ \neq \sigma = \text{cf}(\mu)$ we have \Diamond_S ([Sh 108]). So let $\langle f_\delta : \delta \in S \rangle$ be such that $f_\delta : \delta \rightarrow [\delta]^\kappa$ satisfy $(\forall f : \lambda \rightarrow [\lambda]^\kappa)(\exists^{\text{stat}} \delta \in S)(f_\delta = f \upharpoonright \delta)$. For each $\delta \in S$, let $B_\delta = \{\alpha_{\delta, \varepsilon} : \varepsilon < \sigma^+\}$ increasing with ε and let $g_\delta : \kappa^{++} \rightarrow [\kappa^{++}]^{\leq \kappa}$ be defined by

$$g_\delta(\beta) = \{\gamma < \kappa^{++} : \text{for some } \varepsilon_1, \varepsilon_2 < \sigma^+ \text{ we have } \alpha_{\delta, \varepsilon_1} \times \kappa^{++} + \gamma \in f_\delta(\alpha_{\delta, \varepsilon_2} \times \kappa^{++} + \beta)\}.$$

So by a variant of the Δ -system lemma (or use $(2^\kappa)^+$ instead κ^{++} if we avoid GCH) there is $Z_\delta \in [Z_\delta]^{\kappa^{++}}$ such that $\gamma_1 \neq \gamma_2 \in Z_\delta \Rightarrow \gamma_1 \notin g_\delta(\gamma_2)$. Let $\gamma_{\delta, \varepsilon} \in Z_\delta$ be strictly increasing with $\varepsilon < \sigma^+$ and let $B'_\delta = \{\alpha_{\delta, \varepsilon} \times \kappa^{++} + \gamma_{\delta, \varepsilon} : \varepsilon < \sigma^+\}$. So clauses (A), $(B)_1$ are immediate. Now clearly $(C)^+$ of 2.1 holds hence (C) and $(B)_2$ of 1.2 follow from the assumption on S (see [Sh 108]).

Now $\mathcal{A} = \{B'_\delta : \delta \in S\}$ are as required in Theorem 1.2. $\square_{2.3}$

2.4 Conclusion: If $\text{CON}(\exists \text{ supercompact})$, then $\text{CON}(\text{CH} + \text{there is a } T_3\text{-topological space } X \text{ with clopen basis, even compact, with } \aleph_{\omega+1} \text{ members, } \aleph_{\omega+1} \text{ nodes such that if we divide } X \text{ to countably many parts, at least one contains a closed copy of the Cantor set})$.

Proof. By 2.3 + 1.2. $\square_{2.4}$

Instead of using [HJSh 249], [Sh:F276] we can directly use [Sh 108], recall that there even G.C.H. holds.

* * *

Lastly, we start to resolve the connection between the various statements around. Now [HJSh 249] continue and strengthen [Sh 108], [Sh 88a]. We show that by a “small nice forcing” (not involving extra large cardinals assumption) we can get the result of [HJSh 249] used above from the one in [Sh 108], [Sh 88a]. (See also [Sh 652, §5] on the semi-additive colouring involved).

2.5 Claim. *Assume*

- (a) $cf(\mu) = \kappa < \mu, (\forall \alpha < \mu)(|\alpha|^\kappa < \mu)$
- (b) $S \subseteq \{\delta < \mu^+ : cf(\delta) = \kappa^+\}$ is stationary, $S \notin I[\lambda]$ and
- (c) $2^{\kappa^+} \leq \mu$ and $\kappa = \kappa^{<\kappa}$.

Then for some forcing notion Q we have:

- (a) Q is $(< \kappa)$ -complete, $|Q| = \kappa^+$ and Q is κ^+ -c.c.
- (b) in V^Q , for some stationary $S' \subseteq S$ we have $\langle A_\delta : \delta \in S' \rangle, A_\delta$ an unbounded subset of δ of order type κ^+ and $\delta_1 \neq \delta_2 \in S' \Rightarrow |A_{\delta_1} \cap A_{\delta_2}| < \kappa$.

Proof. Let $\mu = \sum_{i < \kappa} \lambda_i$ where $\lambda_i < \mu$ is increasing continuous with $i, \lambda_0 > \kappa$. Choose $\bar{A} = \langle A_\alpha : \alpha \in S \rangle$, with $A_\alpha = \{\gamma_{\alpha, \varepsilon} : \varepsilon < \kappa^+\}$ any unbounded subset of α of order type κ^+ and $\gamma_{\alpha, \varepsilon}$ increasing with ε .

We can find $\bar{a}^\alpha = \langle a_i^\alpha : i < \kappa \rangle$ for $\alpha < \mu^+$ such that

- (*)₁ $\alpha = \bigcup_{i < \kappa} a_i^\alpha, a_i^\alpha$ is increasing continuous in $i, |a_i^\alpha| \leq \lambda_i$
- (*)₂ if $\alpha \in a_i^\beta$ then $a_i^\alpha \subseteq a_i^\beta$.
Without loss of generality
- (*)₃ $A_\alpha \subseteq a_0^\alpha$.

Let $\mathbf{c} : [\mu^+]^2 \rightarrow \kappa$ be $\mathbf{c}\{\alpha, \beta\} = \text{Min}\{i : \alpha \in a_i^\beta\}$ for $\alpha < \beta < \lambda^+$ so

$$\boxtimes \alpha < \beta < \gamma \Rightarrow \mathbf{c}\{\alpha, \gamma\} \leq \text{Max}\{\mathbf{c}\{\alpha, \beta\}, \mathbf{c}\{\beta, \gamma\}\}.$$

For $\alpha \in S$ let $c_\alpha : [\kappa^+]^2 \rightarrow \kappa$ be defined by:

for $\varepsilon < \zeta < \kappa^+$ we let

$$c_\alpha\{\varepsilon, \zeta\} = \mathbf{c}\{\gamma_{\alpha, \varepsilon}, \gamma_{\alpha, \zeta}\}.$$

Let $\mathcal{C} = \{c_\alpha : \alpha \in S\}$ so $c_\alpha \in ([\kappa^+]^2)^\kappa$, so $|\mathcal{C}| \leq 2^{\kappa^+}$. Let for $c \in \mathcal{C}, S_c = \{\alpha \in S : c_\alpha = c\}$, so $\langle S_c : c \in \mathcal{C} \rangle$ is a partition of S to $\leq 2^{\kappa^+} < \mu^+$ sets hence necessarily for some $c \in \mathcal{C}$ we have

- (*)₄ $S_c \notin I[\lambda]$.

We fix c . We define a forcing notion Q :

- (A) $p \in Q$ iff $p = (u^p, \xi^p)$ where $u^p \in [\kappa^+]^{<\kappa}$ and $\xi^p < \kappa$ and $\text{Rang}(c \upharpoonright [u^p]^2) \subseteq \xi^p$
- (B) $Q \models p \leq q$ iff: $(p, q \in Q)$ and
 - (i) $u^p \subseteq u^q$
 - (ii) $\xi^p \leq \xi^q$
 - (iii) for every $\beta \in u^p$ and $\alpha \in (u^q \setminus u^p) \cap \beta$ we have $\mathbf{c}\{\alpha, \beta\} \geq \xi^p$

(*)₅ (a) Q is a partial order

(b) $Q' = \{p \in Q : u^p \text{ has a maximal element}\}$ is a dense subset of Q

[why? check Clause (a), as for clause (b), for any $p \in Q$ choose $j \in (\sup(u^p) + 1, \kappa^+)$ and define $q = (u^q, \xi^q)$ by $u^q = u^p \cup \{j\}$ and $\xi^q = \xi^p$]

(*)₆ Q satisfies the κ^+ -c.c.

[why? assume toward contradiction that $\langle p_i : i < \kappa^+ \rangle$ are pairwise incompatible. Without loss of generality $p_i \in Q'$. Without loss of generality $\langle u^{p_i} : i < \kappa^+ \rangle$ is a Δ -system with heart u^* . So without loss of generality $\xi^{p_i} = \xi^*$. So $C = \{\delta < \kappa^+ : u^{p_\delta} \setminus u^* \text{ is disjoint to } \delta \text{ and } (\forall j < \delta)(u^{p_j} \subseteq \delta)\}$ is a club of κ^+ . Let for $\delta \in C$, $\varepsilon_\delta = \text{Min}(u^{p_\delta} \setminus \delta)$ and $\zeta_\delta = \max(u^{p_\delta})$ so $\delta \leq \varepsilon_\delta \leq \zeta_\delta$. Now assume $\alpha < \beta$ are from C , and p_α, p_β is incompatible. Why is $q = (u^{p_\alpha} \cup u^{p_\beta}, \zeta)$ not a common upper bound where we let $\zeta = \sup(\{\xi^*\} \cup \text{Rang}(\mathbf{c} \upharpoonright [u^{p_\alpha} \cup u^{p_\beta}]^2)) + 1$? As $q \in Q$ and as $u^{p_\alpha} \cap \alpha = u^{p_\beta} \cap \beta$, $u^{p_\alpha} \subseteq \beta$ and $\xi^* = \xi^{p_\alpha} = \xi^{p_\beta}$ so $p_\alpha \leq q$, hence necessarily $\neg p_\beta \leq q$ so clause (iii) of (B) fails, i.e. for some $\gamma_1 \in u^{p_\alpha} \setminus \alpha$ and $\gamma_2 \in u^{p_\beta} \setminus \beta$ we have $\mathbf{c}\{\gamma_1, \gamma_2\} < \xi^{p_\beta} = \xi^*$. But $\varepsilon_\alpha \leq \gamma_1$ and $\varepsilon_\alpha < \gamma_1 \Rightarrow \mathbf{c}\{\varepsilon_\alpha, \gamma_1\} < \xi^{p_\alpha} = \xi^*$ and $\gamma_2 \leq \zeta_\beta$ and $\gamma_2 < \zeta_\beta \Rightarrow \mathbf{c}\{\gamma_2, \zeta_\beta\} < \xi^{p_\beta} = \xi^*$. Hence by \boxtimes necessarily $\mathbf{c}\{\varepsilon_\alpha, \zeta_\beta\} < \xi^*$. So for $\delta \in S_c$, $\langle \gamma_{\delta, \varepsilon_i} : i \in C \rangle$ is strictly increasing hence with limit δ and for each $i \in C$, $\gamma_{\delta, \varepsilon_i}$ is above $\{\gamma_{\delta, \varepsilon_j} : j < i\}$ but $< \delta$ and

$$j < i \Rightarrow \mathbf{c}\{\gamma_{\delta, \varepsilon_j}, \gamma_{\delta, \varepsilon_i}\} < \xi^* \Rightarrow \gamma_{\delta, \varepsilon_j} \in a_{\xi^*}^{\gamma_{\delta, \varepsilon_i}}.$$

By [Sh 108] it follows that $S \in I[\lambda]$ (or directly, for every $\gamma < \lambda$, $|\{\langle \gamma_{\delta, \varepsilon_j} : j \in C \cap i^* \rangle : \delta \in S, i^* \in C_\zeta, \gamma_{\delta, \varepsilon_i^*} = \gamma\}| < \lambda$ as for each $i < \kappa^+$ (and γ) we have $\leq |a_{\xi^*}^\gamma|^{|\mathbf{c}|} \leq (\lambda_{\xi^*})^{|\mathbf{c}|} \leq \mu$ possibilities); contradiction. So Q satisfies the κ^+ -c.c.]

Now clearly for every $i < \kappa^+$ there is $p_i \in Q'$ such that $i \in u^{p_i}$, hence (by (*)₆), for some $i(*) < \kappa^+$ we have $p_{i(*)} \Vdash_Q "W_1 = \{i : p_i \in G \text{ and } \text{cf}(i) = \kappa\} \text{ is stationary in } \kappa^+"$. Let $p_{i(*)} \in G \subseteq Q$, G generic over V and $W_1 = W_1[G]$. Let $C = \{\delta < \kappa^+ : (\forall i < \delta) \sup(u^{p_i}) < \delta\}$, it is a club of κ^+ . Let $W_2 = C \cap W_1$ and for $i \in S_2$ let $\varepsilon_i = \text{Min}(u^{p_i} \setminus i)$, $\zeta_i = \max(u^{p_i})$. Now

(*)₇ if $i \in W_2$ and $\xi < \kappa$, then $\{j \in W_1 \cap i : \mathbf{c}(\varepsilon_j, \varepsilon_i) < \xi\}$ has cardinality $< \kappa$.

[Why? By density argument for some $q \in G$ we have $p_i \leq q$ and $\xi^q > \xi$. Now if $j \in S_1 \cap i \setminus u^q$ then $p_j \in G$ hence for some $q^+ \in G \subseteq Q$ we have $q \leq q^+$ & $p_j \leq q^+$, so $\varepsilon_j \in u^{q^+} \cap \varepsilon_i$ and as $q \leq q^+$ by the definition of \leq_Q , necessarily $\mathbf{c}(\varepsilon_i, \varepsilon_j) \geq \xi^q > \xi$, as asserted.]

Now define for $\delta \in S_c$, $A'_\delta = \{\gamma_{\delta, \varepsilon} : \varepsilon \in W_2\}$. So A'_δ is an unbounded subset of δ of order type κ^+ .

(*)₈ if $\delta_1 \neq \delta_2$ are from S_c then $A'_{\delta_1} \cap A'_{\delta_2}$ has cardinality $< \kappa$.

[Why? Without loss of generality, let $\delta_1 < \delta_2$, let $\varepsilon(*) \in S_2$ be such that $\delta_1 < \gamma_{\delta_2, \varepsilon(*)}$. Assume toward contradiction that $A = A'_{\delta_1} \cap A'_{\delta_2}$ has cardinality $\geq \kappa$. Recall (by (*)₃) that $\beta \in A \Rightarrow \mathbf{c}\{\beta, \delta_1\} = 0$, letting $\xi^* = \mathbf{c}\{\delta_1, \gamma_{\delta_2, \varepsilon(*)}\}$

we get by \boxtimes that $\beta \in A \Rightarrow \mathbf{c}\{\beta, \gamma_{\delta_1, \varepsilon(*)}\} \leq \max\{\mathbf{c}\{\beta, \delta_1\}, \mathbf{c}\{\delta_1, \gamma_{\delta_1, \varepsilon(*)}\}\} = \max\{0, \xi^*\} = \xi^*$.

So $A^- = \{\varepsilon : \gamma_{\delta_2, \varepsilon} \in A\}$ has cardinality κ and $\varepsilon \in A^- \Rightarrow c\{\varepsilon, \varepsilon(*)\} \leq \xi^*$, contradicting $(*)_7$.

So we are done. $\square_{2.5}$

2.6 Claim. *Assume*

- (A)(i) $\lambda > \kappa > \theta > \sigma \geq \aleph_0$ and $\kappa = \kappa^{<\kappa}$
- (B)₁ $\mathcal{A} \subseteq [\lambda]^\theta$ and $A_1 \neq A_2 \in \mathcal{A} \Rightarrow |A_1 \cap A_2| < \sigma$.

Then for some forcing notion Q

- (a) Q is a strategically $< \kappa$ -complete forcing notion (hence add no new sequence of length $< \kappa$)
- (b) Q is κ^+ -c.c. forcing notion of cardinality $\lambda^{<\kappa}$
- (c) in V^Q , clauses (A)(i), (B)₁ above still hold and (B)₂ from 1.2, i.e.
- (B)₂ \mathcal{A} is κ -free
- (d) if $\lambda, \kappa, \mathcal{A}$ satisfies clause (C) of 1.2 in V , then this still holds in V^Q .

Proof. Let $\mathcal{A} = \{A_\zeta : \zeta < \lambda^*\}$ with no repetitions.

Let Q be the set of $p = (v, v_*) = (v^p, v_*^p)$ such that:

- (a) $v_* \subseteq v \in [\lambda^*]^{<\kappa}$
- (b) there is a list $\langle \zeta(\varepsilon) : \varepsilon < \varepsilon^* \rangle$ of v_* such that for every $\varepsilon < \varepsilon^*$ we have $A_{\zeta(\varepsilon)} \cap \bigcup_{\xi < \varepsilon} A_{\zeta(\xi)}$ has cardinality $< \sigma$; we call $\langle \zeta(\varepsilon) : \varepsilon < \varepsilon^* \rangle$ a witness, also the well ordering on u_*^p it induces is called a witness.

The order is defined by

- $p \leq q$ iff (a) $v_*^p \subseteq v_*^q$ and
- (b) $v^p \setminus v_*^p \subseteq v^q \setminus v_*^q$
- (c) every $\bar{\zeta}$ witnessing $p \in Q$ can be end-extended to $\bar{\zeta}'$ witnessing $q \in Q$.

Define a P -name $\underline{Y} = \cup\{v_*^p : p \in G_Q\}$, $\mathcal{A}' = \{A_\zeta : \zeta \in \underline{Y}\}$. Now

- (*)₁ Q is a partial order
- (*)₂ $|Q| = (\lambda^*)^{<\kappa} \leq (\lambda^{<\theta})^{<\kappa} = \lambda^{<\kappa}$
- (*)₃ any increasingly continuous sequence of members of Q of length $< \kappa$ has a least upper bound.
Hence
- (*)₄ Q is strategically $(< \kappa)$ -complete.

For $p \in Q$ let $u^p = \cup\{A_\zeta : \zeta \in v^p\}$

- (*)₅ for $p \in Q$ we have $u^p \in [\lambda]^{<\kappa}$ and $p \leq q \Rightarrow u^p \subseteq u^q$.

Let $Q' = \{p \in Q : \text{if } \zeta < \lambda^* \text{ and } |A_\zeta \cap u^p| \geq \sigma \text{ then } \zeta \in v^p\}$. For $p \in Q$ let $v_\otimes^p = \{\zeta < \lambda^* : |A_\zeta \cap u^p| \geq \sigma\}$, so:

- (*)₆ (a) $v^p \subseteq v_\otimes^p$ and $p \in Q \Rightarrow |v_\otimes^p| \leq \kappa$ and
- (b) if $(\forall \alpha < \kappa)[|\alpha|^\sigma < \kappa]$ then $p \in Q \Rightarrow |v_\otimes^p| < \kappa$, and
- (c) $p \in Q \Rightarrow p \in Q' \equiv v_\otimes^p = v^p$ and
- (d) Q' is a dense subset of Q if $(\forall \alpha < \kappa)[|\alpha|^\sigma < \kappa]$

[Why? E.g. for clause (d), let $p \in Q$ we choose by induction on $\varepsilon \leq \sigma^+ (< \kappa)$ a condition p_ε such that: $p_0 = p, v_\otimes^{p_\varepsilon} = v_\otimes^p, p_\varepsilon$ is increasingly continuous with ε and $v^{p_{\varepsilon+1}} = \{\zeta < \lambda^* : \zeta \in v^{p_\varepsilon} \text{ or just } |A_\zeta \cap u^p| \geq \sigma\}$. There are no problems and p_{σ^+} is as required as $|A_\zeta \cup u^{p_{\sigma^+}}| \geq \sigma \Rightarrow$ for some $\varepsilon < \sigma^+, |A_\zeta \cap u^{p_\varepsilon}| \geq \sigma \Rightarrow$ for some $\varepsilon < \sigma^+, \zeta \in v^{p_{\varepsilon+1}} \subseteq v^{p_{\sigma^+}}$.]

- (*)₇ if $p \in Q', \zeta \in \lambda^* \setminus v^p$ or just $p \in Q, \zeta \in \lambda^* \setminus v_\otimes^p$ then $p' = (v^p \cup \{\zeta\}, v_\otimes^p \cup \{\zeta\})$ and $p'' = (v^p \cup \{\zeta\}, v_\otimes^p)$ are in Q (even $p \in Q' \Rightarrow p' \in Q'$) and are $\geq p$.

We say $p_0, p_1 \in Q$ are isomorphic if $\text{otp}(v^{p_0}) = \text{otp}(v^{p_1}), \text{otp}(u^{p_0}) = \text{otp}(u^{p_1})$, and $\text{OP}_{v^{p_1}, v^{p_0}}$ maps $v_\otimes^{p_0}$ onto $v_\otimes^{p_1}, \text{OP}_{u^{p_1}, u^{p_0}}$ maps u^{p_0} onto u^{p_1} and for $\zeta \in v^{p_0}, \alpha \in u^{p_0}$ we have $\alpha \in A_\zeta \Leftrightarrow \text{OP}_{u^{p_1}, u^{p_0}}(\alpha) \in A_{\text{OP}_{v^{p_1}, v^{p_0}}(\zeta)}$

- (*)₈ Q satisfies the κ^+ -c.c.

[Why? Let $p_\alpha \in Q$ for $\alpha < \kappa^+$. Let $v_\alpha = \bigcup_{\beta < \alpha} v_\otimes^{p_\beta}$ and $u_\alpha = \bigcup \{A_\zeta : \zeta \in v_\alpha\}$ so $u^{p_\beta} \subseteq u_\alpha$ for $\beta < \alpha$. As $v^{p_\alpha} \in [\lambda^*]^{< \kappa}$, we can find stationary $\{S < \kappa^+ : \text{cf}(\delta) = \kappa\}$ and v such that $\alpha \in S \Rightarrow v^{p_\alpha} \cap v_\alpha = v$. Similarly without loss of generality $\alpha \in S \Rightarrow u^{p_\alpha} \cap u_\alpha = u$. Without loss of generality for $\alpha, \beta \in S$ the conditions p_α, p_β are isomorphic, the isomorphism preserving v and u . So $v_\otimes^{p_\alpha} \cap v = v_\otimes^{p_\beta} \cap v = v_\otimes^{p_\beta}$ for some $v_\otimes^{p_\beta} \subseteq v$. Let $<_\alpha^*$ be a well ordering of $v_\otimes^{p_\alpha}$ which witnesses $p_\alpha \in Q$, so without loss of generality $<_\alpha^* \upharpoonright v_\otimes^{p_\beta} = <_\beta^*$. Let $\alpha < \beta$ be in S and define $q = (v^{p_\alpha} \cup v^{p_\beta}, v_\otimes^{p_\alpha} \cup v_\otimes^{p_\beta})$. Clearly $v_\otimes^q \subseteq v_\otimes^q \in [\lambda^*]^{< \kappa}$, also $\zeta \in v^{p_\alpha} \setminus v^{p_\beta}$ or $\zeta \in v^{p_\beta} \setminus v^{p_\alpha}$ implies $|A_\zeta \cap u| < \sigma$ (why if not we can find $\zeta_1 \in v^{p_\alpha} \setminus v^{p_\beta}, \zeta_2 \in v^{p_\beta} \setminus v^{p_\alpha}$ such that $\zeta \in \{\zeta_1, \zeta_2\}$ and $A_{\zeta_1} \cap u = A_{\zeta_2} \cap u$, so $|A_{\zeta_1} \cap u| \leq |A_{\zeta_1} \cap A_{\zeta_2}| < \sigma$ hence $|A_\zeta \cap u| < \sigma$). Hence $\zeta \in v^{p_\alpha} \setminus v^{p_\beta} \Rightarrow |A_\zeta \cap u^{p_\beta}| < \sigma$ (otherwise $A_\zeta \cap u^{p_\beta} \subseteq u_\beta \cap u^{p_\beta} = u$ hence $|A_\zeta \cap u| \geq \sigma$ and get a contradiction by the previous statement) and $\zeta \in v^{p_\beta} \setminus v^{p_\alpha} \Rightarrow |A_\zeta \cap v^{p_\alpha}| < \sigma$ (similar proof). Now define a two-place relation $<^*$ on v_\otimes^q :

$$\begin{aligned} \zeta_1 <^* \zeta_2 &\text{ iff } \zeta_1 <_\alpha^* \zeta_2 \text{ (so } \zeta_1, \zeta_2 \in v^{p_\alpha}) \\ &\text{ or } \zeta_1 \in v_\otimes^{p_\alpha} \text{ \& } \zeta_2 \in v_\otimes^{p_\beta} \setminus v_\otimes^{p_\alpha} \\ &\text{ or } \{\zeta_1, \zeta_2\} \subseteq v_\otimes^{p_\beta} \setminus v_\otimes^{p_\alpha} \text{ \& } \zeta_1 <_\beta^* \zeta_2 \end{aligned}$$

Easily $<^*$ is a well order of v_\otimes^q (as $\zeta \in v_\otimes^{p_\beta} \setminus v_\otimes^{p_\alpha} \Rightarrow |A_\zeta \cap u^{p_\alpha}| < \sigma$, and it is a witness). So $q \in Q$. Does $p_\alpha \leq q$? Clauses $(\alpha), (\beta)$ are very straight and for clause (γ) , as p_α, p_β are isomorphic for any given witness $<^1$, a well ordering of $v_\otimes^{p_\alpha}$, we can find $<^2$, a witness for p_β which is a well ordering of $v_\otimes^{p_\beta}$, and is conjugate to $<^1$; now use $<^1, <^2$ as we use $<_\alpha^*, <_\beta^*$ above. So really $p_\alpha \leq q$. Similarly $p_\beta \leq q$.]

(*)₉ \Vdash_Q “ $\mathcal{A}' = \{A_\zeta : \zeta \in \cup\{v_*^p : p \in G_Q\}\}$ is $(< \kappa)$ -free”.

[Why? Read the definitions of Q and of being $(< \kappa)$ -free, remembering that forcing with Q add no new sets of ordinals $< \kappa$ as it is strategically $(< \kappa)$ -complete.]

(*)₁₀ if $p, q \in Q$ are compatible, then they have an upper bound $r \in Q$ such that $v^r = v^p \cup v^q$

(*)₁₁ if \mathcal{A} satisfies clause (C) of 1.2 then \mathcal{A}' satisfies this in V^Q .

[Why? Assume $p^* \in Q, p^* \Vdash_Q$ “ $\underline{F} : \lambda \rightarrow [\lambda]^{\leq \kappa}$ is a counterexample”. Without loss of generality $p^* \in Q'$. As Q satisfies the κ^+ -c.c. and as increasing the $\underline{F}(\alpha)$ is O.K., without loss of generality each $\underline{F}(\alpha)$ is from V and $\underline{F} = F$. As we can increase each $F(\alpha)$, without loss of generality $\zeta \in v_\otimes^{p^*} \Rightarrow A_\zeta \subseteq \bigcap_{\alpha} F(\alpha)$. As V, \mathcal{A} satisfies clause (C) there are ζ and $A \in [A_\zeta]^\theta$ which is F -free, by the previous sentence $\zeta \notin v_\otimes^{p^*}$. Define $q = (v^q, v_*^q), v^q = v^{p^*} \cup \{\zeta\}, v_*^q = v_*^{p^*} \cup \{\zeta\}$. It is easy to prove $p^* \leq q \in Q$, the point being $|A_\zeta \cap \{A_\xi : \xi \in v_\otimes^{p^*}\}| < \sigma$ which holds as $\zeta \notin v_\otimes^{p^*}$, and q forces that $A \in [A_\zeta]^\theta$ is as required concerning F . An alternative (older) proof is for each α let $\langle p_{\alpha,i} : i < \kappa \rangle$ be a maximal antichain above p^* of Q such that $p_{\alpha,i} \Vdash_Q$ “ $\underline{F}(\alpha) = a_{\alpha,i}$ ”.

Define $F : \lambda \rightarrow [\lambda]^{\leq \kappa}$ in V by $F(\alpha) = \bigcup_{i < \kappa} a_{\alpha,i} \cup \bigcup_{i < \kappa} u^{p_{\alpha,i}}$. So as in V, \mathcal{A} satisfies clause (C), there is $\zeta \in \mathcal{A}$ such that A_ζ is F -free or just some $A \in [A_\zeta]^\theta$ is F -free. Let $A = \{\gamma_\varepsilon : \varepsilon < \theta\}$. We can now choose by induction $\varepsilon < \theta, (p_\varepsilon, j_\varepsilon)$ such that:

- (a) $p_\varepsilon \in Q$ is increasingly continuous
- (b) $j_\varepsilon < \kappa$
- (c) $p_0 = p^*$
- (d) $p_{\varepsilon+1}$ is an upper bound of $p_\varepsilon, p_{\gamma_\varepsilon, j_\varepsilon}$
- (e) $v^{p_{\varepsilon+1}} = v^{p_\varepsilon} \cup v^{p_{\gamma_\varepsilon, j_\varepsilon}}$
- (f) $v^{p_\varepsilon} = \cup\{v^{p_{\gamma_\xi, j_\xi}} : \xi < \varepsilon\} \cup v^{p^*}$.

For $\varepsilon = 0$ use clause (c), for ε limit use (*)₃ and clause (a) for successor $\varepsilon + 1$ let $j_\varepsilon = \text{Min}\{j : p_\varepsilon, p_{\gamma_\varepsilon, j} \text{ are compatible}\}$ (well defined as $\langle p_{\gamma_\varepsilon, j} : j < \kappa \rangle$ is a maximal antichain above p^*), and define $p_{\varepsilon+1}$ by (*)₁₀.

Let $p_\theta = \bigcup_{\varepsilon < \theta} p_\varepsilon$. Lastly, define q :

$$v^q = v^{p_\theta} \cup \{\zeta\}$$

$$v_*^q = v_*^{p_\theta} \cup \{\zeta\}$$

It suffices to prove: $p^* \leq q \in Q$ and $\varepsilon < \theta \Rightarrow p_{\gamma_\varepsilon, j_\varepsilon} \leq q$ as then q forces A_ζ is as required. The non-trivial part is showing

- ☐ if $<_0^*$ is a witness to $p_{\gamma_\varepsilon, j_\varepsilon}$ then some well ordering $<_1^*$ of v_*^q which end extends it is a witness to q .

First we can find $<_1^*$, a well ordering of $v_*^{p_\theta}$ end extending $<_0^*$ and which is a witness to p_θ . We now define $<^*$, a well ordering of $v_*^q : <^* \restriction v_*^{p_\theta} = <_1^*$, and by $<^*$, ζ is just above all elements of $v^{p_{\gamma_\varepsilon, j_\varepsilon}}$ and below all elements of $v_*^{p_\theta} \setminus v^{p_{\gamma_\varepsilon, j_\varepsilon}}$. Now $<^*$ is as required (note that we have not proved $p_\theta \leq q!$). $\square_{2.6}$

2.7 Observation. Assume that $\kappa = \kappa^{<\kappa} < \lambda$ and $S \subseteq \lambda$ stationary. Then for some κ^+ -c.c., strategically κ -complete forcing notion Q of cardinality $\lambda^{<\kappa}$, we have \Vdash_Q “ S is the union of $\leq \kappa$ sets each not reflecting any δ of cofinality $\leq \kappa$ ”.

Proof. Straightforward.

§3 EQUI-CONSISTENCY

The following theorem clarifies the consistency strength of the problem to a large extent. We can hardly expect more with no inner models for supercompacts. Concentrating on ${}^\omega 2$ is for historical reason; we can replace \aleph_0 by μ . Also, using the same claims we can replace $\lambda > \beth_2$ by other restrictions. Note 3.5 continues [Sh 460, §3], [HJSh 249]. The claims will give more, naturally. However:

3.1 Problem: What occurs if we demand GCH?

3.2 Theorem. *The following are equi-consistent (with $ZFC + \kappa = \text{cf}(\kappa) > 2^{\aleph_0}$ (in fact we use only forcing which preserves the cardinals $\leq (2^{\aleph_0})^+$ and do not change the value of 2^{\aleph_0} , in fact the composition of κ -complete and c.c.c. of cardinality $\leq 2^{\aleph_0}$ ones; so we can add $2^{\aleph_0} = \aleph_1$ or $2^{\aleph_0} = \aleph_2$ or $2^{\aleph_0} = \aleph_{\omega^3 + \omega + 3}$ or whatever)*

- $(A)[{}^\omega 2] = (A)_{({}^\omega 2)}$ *there is a compact Hausdorff space X such that $X \rightarrow_w ({}^\omega 2)_2^1$ but no subspace with $\leq 2^{<\kappa}$ points has this property (on \rightarrow_w see 1.1(2) and ${}^\omega 2$ is the Cantor discontinuum)*
- $(A)^+$ *like $(A)_{({}^\omega 2)}$ replacing ${}^\omega 2$ by “for any Hausdorff space Y^* with $\leq 2^{\aleph_0}$ points” and demand X has a clopen basis only if Y has*
- $(B)[{}^\omega 2] = (B)_{({}^\omega 2)}$ *there is a compact Hausdorff space X with clopen basis such that $X \rightarrow ({}^\omega 2)_{<\text{cf}(2^{\aleph_0})}^1$ but no subspace with $\leq 2^{<\kappa}$ points has this property*
- $(B)^+$ *like $(B)[{}^\omega 2]$ replacing ${}^\omega 2$ by “for any Hausdorff space with $\leq 2^{\aleph_0}$ points” and demand X has a clopen basis only if Y has*
- (C) *there are λ, S, \bar{f} such that*
 - (a) $S \subseteq \lambda$ is stationary, $\lambda > 2^{<\kappa}$ is regular
 - (b) $\bar{f} = \langle f_\delta : \delta \in S \rangle$
 - (c) f_δ is a one-to-one function from ${}^\omega 2$ to δ
 - (d) if $\delta_1 \neq \delta_2$ then $\{\eta \in {}^\omega 2 : f_{\delta_1}(\eta) = f_{\delta_2}(\eta)\}$ has scattered closure (in the topological space ${}^\omega 2$)
- (D) *there are λ, S, \bar{A} such that*
 - (a) $S \subseteq \lambda$ is stationary, $\lambda > 2^{<\kappa}$ is regular
 - (b) $\bar{A} = \langle A_\delta : \delta \in S \rangle$
 - (c) A_δ is a subset of δ of cardinality 2^{\aleph_0}
 - (d) for $\delta_1 \neq \delta_2$ from S we have $A_{\delta_2} \cap A_{\delta_1}$ is finite
 - (e) $\{A_\delta : \delta \in S\}$ is κ -free
 - (f) if $F : \lambda \rightarrow \alpha^*, \alpha^* < \lambda$ then for some $\delta, F \restriction A_\delta$ is constant.

Note that we can easily add clauses sandwiched between two existing ones.

Question: With what can we replace the space ${}^\omega 2$?

We make some definitions and prove some claims before the proof. The following definition is used in 3.5.

3.3 Definition. 1) For a cardinal κ and $I_0, I_1 \subseteq \{(a, b) : a, b \subseteq \kappa \text{ are disjoint}\}$ and cardinal θ we say that a cardinal λ is (I_0, I_1, θ) -approximate or $(\kappa, I_0, I_1, \theta)$ -approximate if we can find $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha \in C \rangle$ such that

- (i) C a club of λ
- (ii) $\mathcal{P}_\alpha \subseteq [\alpha]^{<\theta}$ for $\alpha \in C$ and $|\mathcal{P}_\alpha| \leq \text{Min}(C \setminus (\alpha + 1))$
- (iii) for any 1-to-1 function f from κ to λ , for some $\alpha \in C$ one of the following holds
 - (a) for some $c \in \mathcal{P}_\alpha$ and $(a, b) \in I_1$ we have $(\forall i \in a)(f(i) \in c)$ and $(\forall i \in b)[f(i) \geq \alpha]$
 - (b) for some $(a, b) \in I_0$ we have

$$(\alpha) \quad (\forall i < \kappa)(f(i) < \alpha \rightarrow i \in a)$$

$$(\beta) \quad (\forall i < \kappa)[i \in b \rightarrow \alpha \leq f(i) < \text{Min}(C \setminus (\alpha + 1))]$$

2) If $c\ell$ is a function from $\mathcal{P}(\kappa)$ to $\mathcal{P}(\kappa)$ and $K \subseteq \mathcal{P}(\kappa)$ and

$$I_1 = \{(a, b) : a \subseteq \kappa, b \in K \text{ and } b \subseteq c\ell(a)\}$$

$$I_0 = \{(a, b) : a \subseteq \kappa, b \in K \text{ and } a \cap b = \emptyset\}$$

then we may say λ is $(K, c\ell, \theta)$ -approximate or $(\kappa, K, c\ell, \theta)$ -approximate instead of λ is (I_0, I_1, θ) -approximate.

3) We may replace κ by another set of this kind call the domain of the tuple (understood from I_0, I_1). We may write this set before I_0 for clarification.

4) We may replace (I_0, I_1, θ) by (\mathbf{I}, θ) if \mathbf{I} is a set of pairs (I_0, I_1) such that $\langle \mathcal{P}_\alpha : \alpha \in C \rangle$ satisfies the requirement above all the triples (I_0, I_1, θ) such that $(I_0, I_1) \in \mathbf{I}$ (not necessarily all pairs have the same domain A).

Similarly, \mathbf{K} stands for a set of tuples $(\kappa, K, c\ell, \theta)$ or in short $(\kappa, K, c\ell)$ when θ is understood from the context or even $(K, c\ell)$ as in part (2). (We may even vary θ).

Concerning 4.4 below

3.4 Examples: 1) Let \mathbf{C} be a Cantor set (say ${}^\omega 2$),

$c\ell^{\mathbf{C}}$ is the (topological) closure operation on subsets of \mathbf{C}

$K^{\mathbf{C}} = \{A \subseteq \mathbf{C} : A \text{ is closed perfect uncountable}\}.$

2) Let \mathbb{R} be the real line, $c\ell^{\mathbb{R}}$ be the (topological) closure operation on subsets of \mathbb{R}

and $K^{\mathbb{R}} = \{A \subseteq \mathbb{R} : A \text{ is closed perfect uncountable, bounded (from below and above)}\}.$

3.5 Lemma. *Assume*

- (a) $\lambda > \chi \geq \kappa \geq \theta$ and σ are infinite cardinals,
- (b) cl is a partial function from $[\lambda]^{<\theta}$ to $K \subseteq [\lambda]^{\leq \kappa}$
- (c) \mathbf{K} is a set of triples (κ, K^*, cl^*) with $K^* \subseteq \mathcal{P}(\kappa)$, cl^* a function from $[\kappa]^{<\theta}$ to $\mathcal{P}(\kappa)$ as in Definition 3.3(2) above (for θ)
- (d) if $b \in K$, then for some $(\kappa, K^*, cl^*) \in \mathbf{I}$ and one to one function f from κ into b , we have:

$$b' \in K^* \Rightarrow \{f(\alpha) : \alpha \in b'\} \in K$$

$$a', b' \subseteq \kappa \ \& \ cl^*(a') = b' \Rightarrow cl\{f(\alpha) : \alpha \in a'\} \supseteq \{f(\alpha) : \alpha \in b'\}$$

- (e) for every $A \in [\lambda]^{\leq \chi}$ we can find a $[K, \sigma]$ -colouring \mathbf{c} of A , which is defined for any $A \subseteq \lambda$ as saying that: \mathbf{c} is a function from A to σ such that $a \in K$ & $a \subseteq A \Rightarrow \text{Rang}(\mathbf{c} \upharpoonright a) = \sigma$
- (f) for every μ , if $\chi < \mu \leq \lambda$ then μ is (\mathbf{K}, θ) -approximate.

Then there is $[K, \sigma]$ -colouring \mathbf{c} of λ .

Proof. See after the proof of 4.12 below. (The reader may prefer to read first §4 up to the proof of 3.5, 3.11 first).

3.6 Conclusion: 1) Assume

- (a) every cardinal $\mu, 2^{\aleph_0} < \mu \leq \lambda$ is $(\mathbf{C}, K^{\mathbf{C}}, cl^{\mathbf{C}}, \aleph_1)$ -approximate
- (b) X is a Hausdorff topological space.

Then $X \rightarrow [\text{Cantor set}]_{2^{\aleph_0}}^1$.

2) We can replace in part (1), \mathbf{C} by \mathbb{R} .

Proof. By 3.5 (and 3.4). □_{3.6}

3.7 Claim. *The forcing notions in 1.2 and in 2.5 satisfies e.g. the condition $*_{\kappa^+}^{\sigma^+}$; see below Definition 3.8(1A).*

3.8 Definition. 1) Let D be a normal filter on μ^+ to which $\{\delta < \mu^+ : \text{cf}(\delta) = \mu\}$ belongs. A forcing notion Q satisfies $*_D^\epsilon$ where ϵ is a limit ordinal $< \mu$, if player I has a winning strategy in the following game $*_D^\epsilon[Q]$ defined as follows:

Playing: the play finishes after ϵ moves.

In the ζ -th move:

Player I — if $\zeta \neq 0$ he chooses $\langle q_i^\zeta : i < \mu^+ \rangle$ such that $q_i^\zeta \in Q$

and $(\forall \xi < \zeta)(\forall^D i < \mu^+) p_i^\xi \leq q_i^\zeta$ and he chooses a function $f_\zeta : \mu^+ \rightarrow \mu^+$ such that for a club of $i < \mu^+$, $f_\zeta(i) < i$;

if $\zeta = 0$ let $q_i^\zeta = \emptyset_Q$, f_ζ is identically zero.

Player II — he chooses $\langle p_i^\zeta : i < \mu^+ \rangle$ such that $(\forall^D i) q_i^\zeta \leq p_i^\zeta$ and $p_i^\zeta \in Q$.

The Outcome: Player I wins provided that for some $E \in D$: if

$\mu < i < j < \mu^+, i, j \in E, cf(i) = cf(j) = \mu$ and $\bigwedge_{\xi < \epsilon} f_\xi(i) = f_\xi(j)$ then the set

$\{p_i^\zeta : \zeta < \epsilon\} \cup \{p_j^\zeta : \zeta < \epsilon\}$ has an upper bound in Q .

1A) If D is $\{A \subseteq \mu^+ : \text{for some club } E \text{ of } \mu^+ \text{ we have } i \in E \ \& \ cf(i) = \mu \Rightarrow i \in A\}$ we may write μ instead of D (in $*_D^\epsilon$ and in the related notions defined below and above).

2) We may allow the strategy to be non-deterministic, e.g. choose not f_ζ just f_ζ/D_{μ^+} .

3) We say a forcing notion Q is ϵ -strategically complete if for the following game, \otimes_Q^ϵ player I has a winning strategy.

In the ζ -th move:

Player I - if $\zeta \neq 0$ he chooses $q_\zeta \in Q$ such that $(\forall \xi < \zeta) p_\xi \leq q_\zeta$ if $\zeta = 0$ let $q_\zeta = \emptyset_Q$.

Player II - he chooses $p_\zeta \in Q$ such that $q_\zeta \leq p_\zeta$.

The Outcome: In the end Player I wins provided that he always has a legal move.

3.9 Lemma. *The property “ Q is $(< \mu)$ -strategically complete and has $*_\mu^\epsilon$ ” is preserved by $(< \kappa)$ -support iteration (see [Sh 546]). FILL?*

Proof. Straight; in each coordinate we preserve that the sequence of conditions is increasingly continuous and on each stationary $S \subseteq \{\delta < \kappa^+ : cf(\delta) = \kappa\}$ on which the pressing down function is constant the conditions form a Δ -system. $\square_{3.9}$

We can also consider

3.10 Definition. 1) We say $X^* \rightarrow [Y^*]_\theta^n$ if X^*, Y^* are topological spaces and for every $h : [X^*]^n \rightarrow \theta$ there is a closed subspace Y of X^* homeomorphic to Y^* such that for some $\alpha < \theta, \alpha \notin \text{Rang}(h \upharpoonright [Y]^n)$ is not θ .

2) If we omit the “closed” we shall write \rightarrow_w instead of \rightarrow and $\nrightarrow, \nrightarrow_w$ denote the negations. [FILL? 3.2, 3.5.]

3.11 Claim. 1) Assume X is a Hausdorff space with λ points and D is a filter on ${}^\omega 2$ containing the co-countable subsets of ${}^\omega 2$. Assume further $X \nrightarrow [{}^\omega 2]_\theta^1$ and $\chi \geq 2^{\aleph_0}$ but no subspace X^* of X with $< \chi$ points satisfy $X^* \nrightarrow ({}^\omega 2)_{2^{\aleph_0}}^1$ and $\chi = \chi^{\aleph_0}$. Then

(*) we can find a regular $\kappa \in (\chi, \lambda]$, a stationary $S \subseteq \kappa$ and $\bar{f} = \langle f_\alpha : \alpha \in S \rangle$ such that:

- (i) $\text{Dom}(f_\alpha) \in D^+$
- (ii) f_α is one-to-one and is a homeomorphism from ${}^\omega 2 \upharpoonright \text{Dom}(f_\alpha)$ onto $X \upharpoonright \text{Rang}(f_\alpha)$
- (iii) if $\alpha \neq \beta$ are from S , then $\{\eta \in \text{Dom}(f_\alpha) : f_\alpha(\eta) \in \text{Rang}(f_\beta)\}$ has scattered closure in ${}^\omega 2$
- (iv) for a club of $\delta \in S$ we have $\text{Rang}(f_\alpha) \subseteq \bigcup_{\beta \in \alpha \cap S} \text{Rang}(f_\beta)$.

2) Similarly for \rightarrow_w .

We shall prove it later (after the proof of 4.12).

3.12 Observation: There is a c.c.c. forcing notion Q of cardinality 2^{\aleph_0} such that:

- \Vdash_Q “there is $h : {}^\omega 2 \rightarrow \omega$ such that :
- (α) if $C \in V$ is closed scattered then each $C \cap h^{-1}\{n\}$ is finite, and
 - (β) if $A \subseteq ({}^\omega 2)$ is uncountable and from V then $|A \cap h^{-1}\{n\}| = |A|$ for each n ”.

Proof. Let $p \in Q$ be (f^p, \mathcal{C}^p) where f^p is a finite function from ${}^\omega 2$ to ω and \mathcal{C}^p is a finite family of closed scattered subsets of ${}^\omega 2$.

Order is:

$p \leq q$ iff $f^p \subseteq f^q, \mathcal{C}^p \subseteq \mathcal{C}^q$ and $C \in \mathcal{C}^p$ & $\eta \in C \cap \text{Dom}(f^p)$ & $\nu \in \text{Dom}(f^q) \setminus \text{Dom}(f^p)$ & $\eta \neq \nu \rightarrow f^q(\eta) \neq f^q(\nu)$.

Clearly

(*)₁ Q is a forcing notion of cardinality 2^{\aleph_0}

(*)₂ Q satisfies the c.c.c.

[why? let $p_\alpha \in Q$ for $\alpha < \omega_1$, let $\text{Dom}(f_\alpha) = \{\eta_{\alpha,\ell} : \ell < \ell_\alpha\}$, $\mathcal{C}^{p_\alpha} = \{C_{\alpha,k} : k < k_\alpha\}$ and let $m_\alpha = \text{Min}\{m : \langle \eta_{\alpha,\ell} \restriction m : \ell < \ell_\alpha \rangle$ is with no repetitions}. Without loss of generality $m_\alpha = m(*)$, $\ell_\alpha = \ell(*)$, $k_\alpha = k(*)$, $\eta_{\alpha,\ell} \restriction m(*) = \nu_\ell$. By Δ -system lemma without loss of generality for some $\ell(**) \leq \ell(*)$ we have:

- (α) $\ell < \ell(**) \Rightarrow \langle \eta_{\alpha,\ell} : \alpha < \omega_1 \rangle$ is with no repetitions
- (β) $\ell \in [\ell(**), \ell(*)] \Rightarrow \eta_{\alpha,\ell} = \eta_\alpha$
- (γ) $\{\eta_{\alpha,\ell} : \alpha < \omega_1, \ell < \ell(**)\}$ is with no repetitions.

Now as each $C_{\alpha,k}$ is closed and scattered it is necessarily countable so without loss of generality

$$\alpha < \beta < \omega_1 \text{ \& \> } \ell < \ell(**) \Rightarrow \eta_{\beta,\ell} \notin \bigcup_{k < k(*)} C_{\alpha,k}.$$

We now choose by induction on $\ell < \ell(**)$ sets $A_\ell, B_\ell \in [\omega_1]^{\aleph_1}$, decreasing with n such that

$$\alpha \in A_{\ell+1} \text{ \& \> } \beta \in B_{\ell+1} \text{ \& \> } \alpha < \beta \rightarrow \eta_{\alpha,\ell} \notin \bigcup_{k < k(*)} C_{\beta,k}.$$

This is straight: let $A_\gamma = \omega_1 = B_0$; if A_ℓ, B_ℓ are given, then for some $\alpha_\ell^* \in A_\ell$ the set $\{\eta_{\alpha,\ell} : \alpha \in A_\ell \setminus \alpha_\ell^*\}$ is \aleph_1 -dense in itself, i.e. $(\forall \alpha \in A_\ell \setminus \alpha_\ell^*)(\forall n < \omega)(\exists^{\aleph_1} \beta \in A_\ell)(\eta_{\beta,\ell} \upharpoonright n = \eta_{\alpha,\ell} \upharpoonright n)$, let $T_\ell = \{\eta_{\alpha,\ell} \upharpoonright n : \alpha \in A_\ell \setminus \alpha_\ell^*\}$. So for each $\beta \in B_\ell$ for some $\nu_\beta^\ell \in T_\ell$ we have $(\forall \rho \in \bigcup_k C_{\beta,k})(\neg \nu_\beta^\ell \triangleleft \rho)$ so for

some $\nu_\ell \in T_\ell$ we have $B_{\ell+1} =: \{\beta \in B_\ell : \nu_\beta^\ell = \nu_\ell\}$ is uncountable and let $A_{\ell+1} = \{\alpha \in A_\ell : \nu_\ell \triangleleft \eta_{\alpha,\ell}\}$.

For $\alpha < \beta, \alpha \in A_{\ell(**)}, \beta \in B_{\ell(**)}$, we have p_α, p_β are compatible.]

- (*)₃ if $A \subseteq {}^\omega 2$ is uncountable and $n < \omega$ then $\mathcal{I}_{A,n} =: \{p : \text{for some } \eta \in A, f^p(\eta) = n\}$ is dense open
[why? as $C^* = \cup\{C : C \in \mathcal{C}^p\}$ is closed and scattered hence countable clearly for some $\eta \in A$ we have $\eta \notin C^*$ so $q = (f^p \cup \{(\eta, n)\}, \mathcal{C}^p)$ satisfies $p \leq q \in Q \cap \mathcal{I}_{A,n}$.]
- (*)₄ for each $\eta \in {}^\omega 2$ the set $\mathcal{I}_\eta = \{p : \eta \in \text{Dom}(f^p)\}$ is dense open
[why? for $p \in Q$ let $n = \sup(\text{Rang}(f^p)) + 1$ and letting $q = (f^p \cup \{(\eta, n)\}, \mathcal{C}^p)$ satisfies $p \leq q \in \mathcal{I}_\eta$.]
- (*)₅ for each closed scattered C , the set $\mathcal{I}_C = \{p : C \in \mathcal{C}^p\}$ is dense open
[why? immediate as $p \in Q \Rightarrow p \leq (f^p, \mathcal{C}^p \cup \{C\}) \in Q$.]

Let $\bar{f} = \cup\{f^p : p \in \bar{G}\}$

- (*)₆ \bar{f} is a function from $({}^\omega 2)^v$ to ω and for each closed scattered $C \in V, \bar{f} \upharpoonright C$ is one to one except on a finite set
[why? easy].

Together we are done. □_{3.12}

Proof of 3.2.

$(B)^+ \Rightarrow (B)^{[\omega 2]}$

Trivial (special case).

$(A)^+ \Rightarrow (A)^{[\omega 2]}$

Trivial (a special case).

$(B)^+ \Rightarrow (A)^+$

Trivial (stronger demands).

$(B)^{[\omega 2]} \Rightarrow (A)^{[\omega 2]}$

Trivial (stronger demands).

$(A)^{[\omega 2]} \Rightarrow (C)$

By 3.11 for the filter D , using the monotonicity of $X \rightarrow ({}^\omega 2)_\theta^1$ in $\theta = \{2^\omega \setminus X : X \subseteq {}^\omega 2 \text{ and the closure of } X \text{ in } {}^\omega 2 \text{ is scattered (equivalently countable)}\}$.

$(C) \Rightarrow (D)$

Forcing by $\text{Levy}(\kappa, 2^{<\kappa})$ change nothing so without loss of generality $\kappa = \kappa^{<\kappa}$. Let $\mu, S, \bar{f} = \langle f_\alpha : \alpha \in S \rangle$ as there. Next let Q be the forcing notion from 3.12 so we get the conclusion of 3.12 for the ideal $[A]^{<\aleph_0}$ where $A \subseteq ({}^\omega 2)^V$ has cardinality 2^{\aleph_0} . So letting $A_\alpha = \text{Rang}(f_\alpha \upharpoonright A)$ we get: $A_\alpha \subseteq \alpha, |A_\alpha| = 2^{\aleph_0}$ and for $\alpha \neq \beta$ from

S , $A_\alpha \cap A_\beta$ is finite. So clauses (a)-(d) of (D) holds. Then we force by $\text{Levy}(\lambda, 2^{<\lambda})$ nothing changes but we get \diamond_S . By 2.7 without loss of generality S does not reflect in ordinal δ if $\text{cf}(\delta) \leq \kappa$. So $(*)$ of 2.2 holds hence by 2.3 we get the κ -freeness (clause (e) of $(D) \equiv (B)_2$ of 1.2) and clause (f) (= clause (C) of 1.2).
 [Question: 2.6 not needed.]

$(D) \Rightarrow (B)^+$

We do it by forcing but for the proof any κ such that $\aleph_1 \leq \text{cf}(\kappa) = \kappa, 2^{<\kappa} < \lambda$ can serve.

If $\kappa > 2^{\aleph_0}$ (as in the main case) and we restrict ourselves to spaces Y^* with a basis of cardinality $< \kappa$, then we can use a product instead of iteration for $P_{i(*)}/P_2$ below and proof is easier. By forcing by $\text{Levy}(\lambda, 2^{<\lambda})$ (see 4.3) without loss of generality \diamond_S for the S of (D), this will be preserved for any forcing notion P if P has density $\leq \lambda$, which is our case. Now we use iterated forcing $\langle P_j, Q_i : j \leq i(*), i < i(*) \rangle$ with $(< \kappa)$ -support, each satisfying the $*_{\kappa+}^{\sigma+}$ version of κ^+ -c.c. (see 3.7). Now let Q_0 be as $\text{Levy}(\kappa, 2^{<\kappa})$ and each Q_{1+i} be as in 1.2 for some Y_{1+i}^* (a P_i -name of a topological space as in 1.2) and it forces an example X_{1+i}^* . With suitable book-keeping (if $\kappa > 2^{\aleph_0}$ is easier) we finish as those iterations preserve “ $i < \kappa$ -strategic completeness hence no new set of ordinals of cardinality $< \kappa$ and (the strong version of) κ^+ -c.c.” is preserved.

Still we have to prove that the example X_{1+i}^* we force to satisfy “ $X_{1+i}^* \rightarrow (Y_{1+i}^*)_\sigma^1$ ” has this property not only in $V^{P_{1+i+1}}$ but also in $V^{P_{i(*)}}$. For this we repeat the relevant part of the proof of 1.2 together with the preservation of $(*)_{\kappa+}^{\sigma+}$. $\square_{3.2}$

§4 HELPING EQUICONSISTENCY

4.1 Definition. 1) Let \mathcal{Y} denote a set of pairs of the form (I, J) where $I \subseteq J$ are ideals over a common set called $\text{Dom}(I, J) = \text{Dom}((I, J))$. Let $\kappa(\mathcal{Y}) = \sup\{|\text{Dom}(I, J)| : (I, J) \in \mathcal{Y}\}$. We call \mathcal{Y} standard if for each $(I, J) \in \mathcal{Y}$, the set $\text{Dom}(I, J)$ is a cardinal.

2) $\text{NFr}_1(\lambda, \mathcal{Y})$ if for some $\lambda^* > \lambda$ we have $\text{NFr}_1(\lambda^*, \lambda, \mathcal{Y})$ which means $\lambda \geq |\mathcal{Y}| + \kappa(\mathcal{Y})$ and there are $\langle \mathcal{F}_{(I, J)} : (I, J) \in \mathcal{Y} \rangle$ exemplifying it which means:

- (a) $\mathcal{F}_{(I, J)} \subseteq \{f : f \text{ a function, } \text{Dom}(f) \in J^+\}$
- (b) if $f \neq g \in \mathcal{F}_{(I, J)}$ then $\{x : x \in \text{Dom}(f) \cap \text{Dom}(g) \text{ but } f(x) \neq g(x)\}$ belong to I
- (c) $\lambda \geq |\cup \{\text{Rang}(f) : f \in \mathcal{F}_{(I, J)} \text{ and } (I, J) \in \mathcal{Y}\}|$
- (d) $\lambda < \lambda^* = \sum\{|\mathcal{F}_{(I, J)}| : (I, J) \in \mathcal{Y}\}$.

2) $\text{NFr}_2(\lambda, \mathcal{Y})$ if λ is regular $> |\mathcal{Y}| + \theta(Y)$ and there are $(I, J) \in \mathcal{Y}$ and $\langle f_\delta : \delta \in S \rangle$ such that

- (a) $S \subseteq \lambda$ is stationary
- (b) $\text{Dom}(f_\delta) \in J^+, \text{Rang}(f_\delta) \subseteq \delta$
- (c) $\delta_1 \neq \delta_2 \Rightarrow \{x : x \in \text{Dom}(f_{\delta_1}) \cap \text{Dom}(f_{\delta_2}) \text{ and } f_{\delta_1}(x) = f_{\delta_2}(x)\} \in I$.

3) We omit N from NFr in parts (1) and (2) for the negation.

4.2 Fact: 1) $\text{NFr}_1(\lambda, \mathcal{Y})$ is preserved by increasing \mathcal{Y} to \mathcal{Y}' when $|\mathcal{Y}'| + \kappa(\mathcal{Y}') < \lambda$. Also $\text{NFr}_2(\lambda, \mathcal{Y})$ is preserved by increasing \mathcal{Y} . Similarly if $\text{NFr}_1(\lambda^*, \lambda, \mu, \mathcal{Y}), \lambda^* \geq \lambda_1^* > \lambda_1 \geq \lambda, \lambda_1 \geq |\mathcal{Y}_1| + \theta(\mathcal{Y}_1)$ and $\mathcal{Y}_1 \supseteq Y$ then $\text{NFr}_1(\lambda_1^*, \lambda_1, \mathcal{Y}_1)$.

2) $\text{NFr}_1(\lambda, \mathcal{Y})$ is equivalent to $\text{NFr}_1(\lambda^+, \lambda, \mathcal{Y})$.

3) If λ^* is regular or at least $\text{cf}(\lambda^*) > |\mathcal{Y}|$ and $\text{NFr}_1(\lambda^*, \lambda, \mathcal{Y})$, then there is $(I, J) \in \mathcal{Y}$ such that $\text{NFr}_1(\lambda^*, \lambda, \mu, \{(I, j)\})$.

4) $\text{NFr}_1(\lambda, \mathcal{Y})$ implies $\text{NFr}_2(\lambda^+, \mathcal{Y})$.

Proof. Check.

4.3 Claim. Assume $\text{NFr}_2(\lambda, \{(I, J)\})$ and let $\bar{A} = \langle A_\delta : \delta \in S \rangle$ exemplifies it and let $\mu = (\text{Dom}(I, J))$ and assume $\mu^{++} < \lambda$.

1) If \diamond_S then we can find $\langle A'_\delta : \delta \in S \cap E \rangle$ exemplifying $\text{NFr}_2(\lambda, \{(I, J)\})$, E a club of λ such that

- (*) if $\tau^+ < \lambda, F : \lambda \rightarrow [\lambda]^{<\tau}$ then for some $\delta \in S \cap E$ the set A_δ is F -free (i.e. $\alpha \neq \beta \in A_\delta \Rightarrow \beta \notin F(\alpha)$).

2) The forcing of adding a Cohen subset of λ (i.e. $(\lambda^{>2}, \triangleleft)$) preserve “ \bar{A} exemplifies $\text{NFr}_2(\lambda, \{(I, J)\})$ ” (as it preserves “ S is stationary”), add no bounded subsets to λ and forces \diamond_S .

Proof. 1) As in 2.1.

2) Straight.

□_{4.3}

* * *

Now we give sufficient conditions for the existence of colouring.

4.4 Claim. *Assume:*

- (a) \mathcal{Y} is as in Definition 4.1(1)
- (b) $\lambda > \mu \geq |\mathcal{Y}| + \kappa(\mathcal{Y})$
- (c) for no regular $\kappa \in (\mu, \lambda]$ do we have $NFr_2(\kappa, \mathcal{Y})$
- (d) cl is a function from $[\lambda]^{\leq \mu}$ to $[\lambda]^{\leq \mu}$
- (e) for $A, B \in [\lambda]^{\leq \mu}$ we have $A \subseteq cl(A) = cl(cl(A))$ and $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$
- (f) $\mathcal{P} \subseteq [\lambda]^{\leq \mu}$ satisfies:
 - for every $A \in \mathcal{P}$ there are a pair $(I, J) \in \mathcal{Y}$, a set $\mathcal{U} \in J^+$ and one to one $f : \mathcal{U} \rightarrow A$ such that:
 - (α) if $\mathcal{U}' \subseteq \mathcal{U}$ & $\mathcal{U}' \in I^+$ then for some $A' \in \mathcal{P}$ we have $A' \subseteq A \cap cl(\{f(i) : i \in \mathcal{U}'\})$
 - (β) there are $\mathcal{U}'_\alpha \subseteq \mathcal{U}, \mathcal{U}'_\alpha \in I^+$ for $\alpha < \alpha^*$ for some $\alpha^* \leq \mu$ such that for any $\mathcal{U}' \subseteq \mathcal{U}, \mathcal{U}' \in I^+$ for some $\alpha < \alpha^*$ we have $\mathcal{U}'_\alpha \subseteq \mathcal{U}'$ or at least $cl(\{f(i) : i \in \mathcal{U}'_\alpha\}) \subseteq cl\{f(i) : i \in \mathcal{U}'\}$
 - (γ) $cl(A) = A$.

Then

$Dec(\lambda, \mathcal{P}, \mu, \mathcal{Y})$ for every $\chi > \lambda$ and $x \in \mathcal{H}(\chi)$ there is $\langle M_i : i < \lambda \rangle$ such that:

- (i) $M_\alpha \prec (\mathcal{H}(\chi), \in, <^*_\chi)$
- (ii) $\mu \cup \{\mathcal{Y}, \lambda, \mu, x\} \subseteq M_\alpha$ and $\|M_\alpha\| = \mu$
- (iii) $\bigcup_{\alpha < \lambda} M_\alpha$ includes λ
- (iv) assume $A \in \mathcal{P}$ and define $\alpha(A) = \text{Min}\{\alpha \leq \lambda : \text{if } \alpha < \lambda \text{ then for some } (I, J) \in \mathcal{Y} \text{ and } \mathcal{U} \in J^+ \text{ and } f : \mathcal{U} \rightarrow A \text{ which is one-to-one, we have } \{i \in \mathcal{U} : f(i) \in \bigcup_{\beta \leq \alpha} M_\beta\} \in J^+\}$ and letting $(I, J), \mathcal{U}, f$ be witnesses to $\alpha(A) = \alpha$ we have:
 - $\{i \in \mathcal{U} : f(i) \in M_\alpha \setminus \bigcup_{\beta < \alpha} M_\beta\} \in J^+$
 - moreover for some $X \in M_\alpha$ of cardinality $\leq \mu$ (so $X \subseteq M_\alpha$) we have $\{i \in \mathcal{U} : f(i) \in X \setminus \bigcup_{\beta < \alpha} M_\beta\} \in J^+$
- (v) for any pregiven $\sigma = cf(\sigma) \leq \mu$ we can demand $M_\alpha = \bigcup_{\varepsilon < \sigma} M_{\alpha, \varepsilon}, M_{\alpha, \varepsilon}$ increasing with ε and $\langle M_{\alpha, \zeta} : \zeta \leq \varepsilon \rangle \in M_{\alpha, \varepsilon+1}$ and $\langle M_\beta : \beta < \alpha \rangle \in M_{\alpha, \varepsilon}$.

Now below we shall prove Claim 4.4 follows from the following variant (we change (d), (e), (f)).

4.5 Claim. *Assume*

- (a)' \mathcal{Y} is as in Definition 4.1(1)
- (b)' $|X| = \lambda > \mu \geq |\mathcal{Y}| + \kappa(\mathcal{Y})$
- (c)' for no regular $\kappa \in (\mu, \lambda]$ do we have $NFr_2(\kappa, \mathcal{Y})$
- (d)' $\bar{\mathcal{F}} = \langle \mathcal{F}_t : t \in T \rangle, T$ is a partial order; we consider the \mathcal{F}_t 's as index sets such that $t \neq s \Rightarrow \mathcal{F}_s \cap \mathcal{F}_t = \emptyset$
- (e)' each member $f \in \bigcup_{t \in T} \mathcal{F}_t$ is a function such that for some $(I, J) = (I_f, J_f) \in \mathcal{Y}$ we have $\text{Dom}(f) \in J^+, \text{Rang}(f) \subseteq X$
- (f)' if $t \in T$ and $f \in \mathcal{F}_t$, then there is a subset $T[f]$ of $T_{<f>}$ of cardinality $\leq \mu$ which is a cover which means $(\forall s \in T_{<f>})(\exists t \in T[f])[s \leq_T t]$ where

$$T_{<f>} =: \{r \in T : \text{for some } g \in \mathcal{F}_r \text{ we have } (I_g, J_g) = (I_f, J_f) \text{ and } \{i : i \in \text{Dom}(f), i \in \text{Dom}(g) \text{ and } f(i) = g(i)\} \in I_g^+ = I_f^+\}.$$

THEN

$\text{Dec}(\lambda, \bar{\mathcal{F}}, \mu, \mathcal{Y})$ for every $\chi > \lambda$ and $x \in \mathcal{H}(\chi)$ there is $\langle M_i : i < \lambda \rangle$ such that:

- (i) $M_\alpha \prec (\mathcal{H}(\chi), \in, <_\chi^*)$
- (ii) $\mu \cup \{\mathcal{Y}, \lambda, \mu, x\} \subseteq M_\alpha$ and $\|M_\alpha\| = \mu$
- (iii) $\bigcup_{\alpha < \lambda} M_\alpha$ includes λ
- (iv) if $s \in T$, then for some $t, s \leq_T t \in T$ and for some $\alpha < \lambda$ and $g \in \mathcal{F}_t$ we have
 - (α) $\{i \in \text{Dom}(g) : g(i) \in \bigcup_{\beta < \alpha} M_\beta\} \in J_g$
 - (β) $t, g \in M_\alpha$ hence $\text{Rang}(g) \subseteq M_\alpha$
- (v) for any pregiven $\sigma = \text{cf}(\sigma) \leq \mu$ we can demand $M_\alpha = \bigcup_{\varepsilon < \sigma} M_{\alpha, \varepsilon}$ where $\langle M_{\alpha, \varepsilon} : \varepsilon < \sigma \rangle$ is increasing, $\mu \cup \{Y, \lambda, \mu, M, *\} \subseteq M_{\alpha, \varepsilon}, M_\alpha = \bigcup_{\varepsilon < \sigma} M_{\alpha, \varepsilon}, \langle M_{\alpha, \zeta} : \zeta \leq \varepsilon \rangle \in M_{\alpha, \varepsilon+1}$ and $\langle M_\beta : \beta < \alpha \rangle \in M_{\alpha, \varepsilon}$.

Before proving 4.5 we deduce 4.4 and prepare the ground.

Proof of 4.4 from 4.5. Straight: just let $T = \mathcal{P}$ and for $A \in T$ we let

$$\mathcal{F}_A = \left\{ f : \text{for some } (I, J) \in \mathcal{Y}, \text{Dom}(f) \in J^+, \right. \\ \left. f \text{ is one to one into } A \text{ and } \right. \\ \left. \mathcal{U}' \subseteq \text{Dom}(f) \ \& \ \mathcal{U}' \in I^+ \Rightarrow \text{cl}(\text{Rang}(f \upharpoonright \mathcal{U}')) \in \mathcal{P} \right\}.$$

We define the partial order \leq_T on T by: $A_1 \leq_T A_2$ iff $A_2 \subseteq A_1$. We have to check that the assumptions in 4.5 holds, now clauses $(a)', (b)', (c)'$ are the same as $(a), (b), (c)$ of 4.4, and clauses $(d)', (e)'$ are obvious. As for clause $(f)'$ we shall use clause (f) and the definition of \mathcal{F}_A .

[Why? Let $t \in T, f \in \mathcal{F}_t$ for $t = A \in \mathcal{P}$, let $(I_A, J_A) \in \mathcal{Y}, \mathcal{U} \in J^+$ and f^* be as in clause (f) of 4.4 (with f^* here standing for f there), and let $\langle \mathcal{U}'_\alpha : \alpha < \alpha^* \rangle$ be as in subclause (β) there. For each $\alpha < \alpha^*$ choose A'_α as in subclause (α) of clause (f) of 4.4 for $\mathcal{U}' = \mathcal{U}'_\alpha$. Let us choose $T[f] =: \{A'_\alpha : \alpha < \alpha^*\}$, so $T[f] \in [\mathcal{P}]^{\leq \mu} = [T]^{\leq \mu}$. Let us check that $T[f]$ is as required; being a cover: let $r \in T_{\langle f \rangle}$, i.e. let $A' = r$ and (by the definition of $T_{\langle f \rangle}$), there is $g \in \mathcal{F}_r$ such that $\mathcal{U}' = \{i : i \in \text{Dom}(f) \text{ and } i \in \text{Dom}(g) \text{ and } f(i) = g(i)\} \in I_f^+$, so $r = A'' \in \mathcal{P}$ and $\mathcal{U}' \in I$ so for some $\alpha < \alpha^*$ we have $\text{cl}\{f(i) : i \in \mathcal{U}'_\alpha\} \subseteq \text{cl}\{f(i) : i \in \mathcal{U}'\}$ so by the choice of A'_α we have $A'_\alpha \subseteq \text{cl}\{f(i) : i \in \mathcal{U}'\}$ and $A'_\alpha \in T[f]$, let $s = A'_\alpha$, so $r \in T[f]$ is enough. Now $s \in T_{\langle f \rangle}$ and (by the choice of $s = A'_\alpha$) clearly $A'_\alpha \subseteq A$ which means $r \leq s$. Also $g' = f \upharpoonright \{i \in \text{Dom}(f) : f(i) \in A'_\alpha\}$ belongs to $\mathcal{F}_{A'_\alpha}$ (as $f \in \mathcal{F}_A$) and so g' witness $s \in T[f]$, proving $T[f]$ covers. Lastly, $T[f]$ has cardinality $\leq |\alpha^*| \leq \mu$.]

Lastly let χ be large enough and $x \in \mathcal{H}(\chi)$. So by 4.5 there is $\langle M_i : i < \lambda \rangle$ for our $\langle \mathcal{F}_A : A \in T \rangle, x, \chi$ as required there. It is enough to show that $\langle M_i : i < \lambda \rangle$ is as required in the conclusion of 4.4. Now clauses (i), (ii), (iii) of the conclusion of 4.4 are just like clauses (i), (ii), (iii) and (v) of the conclusion of 4.5, so we should check only clause (iv). So assume $A \in \mathcal{P}$ and $\alpha(A)$ is as defined there. By clause (iv) of the conclusion of 4.5 applied to $s = A$ there are t, α, g as there, i.e. $s \leq_T t, \alpha < \lambda, g \in \mathcal{F}_t$ and $\{i \in \text{Dom}(g) : g(i) \in \bigcup_{\beta < \alpha} M_\beta\} \in J_g$ and $t, g \in M_\alpha$.

So $t \in P, t \subseteq A, \text{Dom}(g) \subseteq t \subseteq s = A, \text{Dom}(g) \subseteq M_\alpha$ and $\text{Rang}(g) \in M_\alpha$. So $\text{Dom}(g) \in J_g^+$ and α is as required. $\square_{4.5}$

4.6 Claim. 1) In 4.4 we can conclude $(\alpha)_\theta \Rightarrow (\beta)_\theta$ when

$(\alpha)_\sigma$ if $\mathcal{P}' \subseteq \mathcal{P}$ has cardinality $\leq \mu$, then we can find $h : \cup\{A : A \in \mathcal{P}'\}$ to θ such that $A \in \mathcal{P}' \Rightarrow \theta = \text{Rang}(h \upharpoonright A)$

$(\beta)_\sigma$ we can find $h : \lambda \rightarrow \theta$ such that $A \in \mathcal{P} \Rightarrow \theta = \text{Rang}(h \upharpoonright A)$.

2) In 4.5 we can conclude $(\alpha)_\theta \Rightarrow (\beta)_\theta$ when

$(\alpha)_\theta$ if $T' \subseteq T, |T'| \leq \mu$ and G is a function with domain $\cup\{\mathcal{F}_t : (\exists s \in T')(s \leq_T t)\}$ such that $G(f) \in J_f$, then we can find a function h and $\langle (t_s, f_s) : s \in T' \rangle$ such that $s \leq_T t_s, f_s \in \mathcal{F}_{t_s}$ and $s \in T_s \Rightarrow \theta = \{(h(f_s(i)) : i \in \text{Dom}(f_s) \setminus G(f_s))\}$

$(\beta)_\theta$ we can find a function $h : \lambda \rightarrow \theta$ as in $(\alpha)_\theta$ for $T' = T$.

Proof. 1) Let $\{A_{\alpha, \zeta}^* : \zeta < \zeta_\alpha \leq \mu\}$ list $\{A \in \mathcal{P} : \alpha(A) = \alpha\}$ and let $(I_\zeta^\alpha, J_\zeta^\alpha), \mathcal{U}_\zeta^\alpha, f_\zeta^\alpha$ witness $\alpha(A_i) = \alpha$. Let $A'_{\alpha, \zeta} = \{f_\zeta^\alpha(i) : i \in \mathcal{U}_\zeta \text{ and } f_\zeta(i) \in M_\alpha \setminus \bigcup_{\beta < \alpha} M_\beta\}$.

Clearly $A'_{\alpha, \zeta} \in \mathcal{P}$ and we apply clause $(\alpha)_\theta$ to $\mathcal{P}_\zeta = \{A'_{\alpha, \zeta} : \zeta < \zeta_\alpha\}$ getting $h_\alpha : \bigcup_{\zeta < \zeta_\alpha} A'_{\alpha, \zeta} \rightarrow \theta$ so without loss of generality $h_\alpha : \lambda \cap M_\alpha \setminus \bigcup_{\beta < \alpha} M_\beta \rightarrow \theta$. Now

$h = \bigcup_{\alpha < \lambda} h_\alpha$ is as required.

2) Similar. □_{4.6}

The following is close to [Sh 161, §3] (or see [Sh 523, §3] or [EM]).

4.7 Definition. 1) We say $\Gamma = (S, \bar{\lambda})$ is a full (λ, μ) -set if:

- (a) S is a set of finite sequences of ordinals
- (b) S is closed under initial segments
- (c) $\bar{\lambda} = \langle \lambda_\eta : \eta \in S \rangle, \lambda_{<} = \lambda$
- (d) for each $\eta \in S$, $\{\alpha : \eta^\frown \langle \alpha \rangle \in S\}$ is empty or the regular $\text{cf}(\lambda_\eta)$
- (e) $\lambda_\eta > \mu$ iff $\lambda_\eta \neq \mu$ iff $(\exists \alpha)(\eta^\frown \langle \alpha \rangle \in S)$ iff $\eta \in S \setminus S^{mx}$
- (f) if $\eta \in S$, $\lambda_\eta \neq \mu$ then for every ordinal α we have $\alpha < \text{cf}(\lambda_\eta) \Leftrightarrow \eta^\frown \langle \alpha \rangle \in S$
- (g) (α) if $\lambda_\eta > \mu$ is a successor cardinal then $\alpha < \lambda_\eta \Rightarrow \lambda_{\eta^\frown \langle \alpha \rangle}^+ = \lambda_\eta$
 (β) if $\lambda_\eta > \mu$ is a limit cardinal then $\langle \lambda_{\eta^\frown \langle \alpha \rangle} : \alpha < \text{cf}(\lambda_\eta) \rangle$
 is strictly increasing with limit λ_η .

2) Let $S^{mx} = \{\eta \in S : \lambda_\eta = \mu\}$.

4.8 Observation/Definition: If $\Gamma = (S, \bar{\lambda})$ is a full (λ, μ) -set, then from S we can reconstruct $\bar{\lambda}$ hence Γ , so we may say “ S is a full (λ, μ) -set, $\bar{\lambda} = \bar{\lambda}^{[S]}$ ”.

4.9 Fact/Definition: 1) If S is a full (λ, μ) -set and $\eta \in S$ let $S^{<\eta>} = \{\nu : \eta^\frown \nu \in S\}$, is a full (λ_η, μ) -set.

2) If for $\alpha < \text{cf}(\lambda)$, S_α is a full (λ_α, μ) set and $(\forall \alpha < \text{cf}(\lambda))(\lambda_\alpha = \lambda_0 \ \& \ \lambda = \lambda_0^+)$ or $\langle \lambda_\alpha : \alpha < \text{cf}(\lambda) \rangle$ is strictly increasing with limit λ , $\lambda_0 \geq \mu$, then $S = \{<>\} \cup \bigcup_{\alpha < \text{cf}(\lambda)} \{\langle \alpha \rangle^\frown \eta : \eta \in S_\alpha\}$ is a full (λ, μ) -set.

3) For a full (λ, μ) -set S and $\eta \in S$, if $\lambda_\eta > \mu$ let $\eta^+ = \langle \eta(\ell) : \ell < k \rangle^\frown \langle \eta(k) + 1 \rangle$ if $\ell g(\eta) = k + 1$, $<>^+$ will be used though not defined.

Proof. Straightforward.

4.10 Definition. 1) We define by induction on λ the following. For a set X of cardinality λ , χ large enough and $x \in \mathcal{H}(\chi)$ we say \bar{N} is a μ -decomposition of X for $\mathcal{H}(\chi), x$ (or (λ, μ) -decomposition) if for some full (λ, μ) -set S it is an S -decomposition of X inside $\mathcal{H}(\chi)$, which means:

- (a) $\bar{N} = \langle (N_\eta, N_\eta^+) : \eta \in S \rangle$
- (b) $N_\eta \prec N_\eta^+ \prec (\mathcal{H}(\chi), \in, <^*)$ except that N_η is an empty set if $\text{Rang}(\eta) \subseteq \{0\}$
- (c) $\{X, x\} \in N_\eta^+$ and $\ell < \ell g(\eta) \Rightarrow N_{\eta \restriction \ell}, N_{\eta \restriction \ell}^+ \in N_\eta^+$
- (d) $\|N_\eta^+\| = \lambda_{\eta^+} = \|(N_\eta^+ \setminus N_\eta) \cap X\|$ and $\lambda_{\eta^+} \subseteq N_\eta^+$
- (e) if $\lambda_{<} > \mu$, then $\langle N_{<\alpha>} : \alpha < \text{cf}(\lambda_{<}) \rangle$ is \prec -increasingly continuous with union containing $N_{<}^+$
- (f) $N_{<\alpha>}^+ = N_{<\alpha+1>}$
- (g) for each $\alpha < \text{cf}(\lambda_{<})$ the sequence $\langle (N_{<\alpha>}^\frown \eta, N_{<\alpha>^\frown \eta}^+) : \eta \in S^{<\alpha>} \rangle$ is a (λ_η, μ) -decomposition of $X \cap N_{<\alpha>}^+$ for $\mathcal{H}(\chi), \langle x, N_\alpha, N_\alpha^+ \rangle$.

2) We say \bar{N} is a (λ, μ, σ) -decomposition of X for $\mathcal{H}(\chi), x$ if $\sigma = \text{cf}(\sigma) \leq \mu$ and in addition

- (h) for each $\eta \in S \setminus S^{\max}$ the sequence $\langle N_{\eta, \varepsilon} : \varepsilon \leq \sigma \rangle$ is increasing continuous, $N_{\eta, 0} = N_\eta$, $\langle N_{\eta, \zeta} : \zeta \leq \varepsilon \rangle \in N_{\eta, \varepsilon+1}$ and the objects we demand $\in N_\eta^+$ belongs to $N_{\eta, 1}$ (in clauses (c) and (h)).

4.11 Definition. 1) Let $X, \lambda, \mu, \mathcal{Y}, \bar{\mathcal{F}}$ be as in 4.5 so $T = \text{Dom}(\bar{\mathcal{F}})$.

We say \bar{N} is a full μ -decomposition of X for x, χ is good for $(X, \mathcal{Y}, \bar{\mathcal{F}})$ if:

- (a) \bar{N} is a full (λ, μ) -decomposition of X for $\mathcal{H}(\chi), \langle x, X, \lambda, \mu, \mathcal{Y}, \mathcal{P} \rangle$; let $\bar{N} = \langle N_\eta : \eta \in S \rangle$ and $\bar{\lambda} = \bar{\lambda}^{[S]}$
- (b) if $s \in T$, then for some $t \in T, s \leq_T t$ and for some $\eta \in S^{mx}$ (i.e. $\lambda_\eta = \mu$) there is $f \in \mathcal{F}_t$ and so $(I_f, J_f) \in \mathcal{Y}, \mathcal{U}_f \in J_f^+, f : \mathcal{U}_f \rightarrow \text{Rang}(g)$ such that:
- (*)₁ $\{i \in \mathcal{U}_f : f(i) \in \cup \{N_\nu : \nu \leq_{\ell_x} \eta \text{ and } \nu \in S^{mx}\}\}$ belongs to J_f
- (*)₂ $\{i \in \mathcal{U}_f : f(i) \in N_\eta^+ \setminus \cup \{N_\nu : \nu <_{\ell_x} \eta \text{ and } \nu \in S^{mx}\}\}$ belongs to J_f^+
- (*)₃ t, f belong to N_η^+ .

2) We define a full (λ, μ, σ) -decomposition similarly.

4.12 Claim. Under the assumption of 4.5, for $x \in \mathcal{H}(\chi), \sigma = \text{cf}(\sigma) \leq \mu$ and χ large enough there is a (λ, μ, σ) -decomposition of X for χ, x good for $(X, \mathcal{Y}, \bar{\mathcal{F}})$.

Proof. By induction on $\lambda = |X|$.

Case 1: $\lambda = \mu$.

Trivial.

Case 2: $\lambda = \text{cf}(\lambda) > \mu$.

Choose $\langle N_\alpha : \alpha < \text{cf}(\lambda) \rangle$ such that $\{x, X, \bar{\mathcal{F}}, \mu, \lambda\} \in N_0, N_\alpha \prec (\mathcal{H}(\chi), \in, <_\chi^*), N_\alpha$ is \prec -increasingly continuous, $\langle N_\beta : \beta \leq \alpha \rangle \in N_{\alpha+1}$, each N_α has cardinality $< \lambda$ and $N_\alpha \cap \lambda$ is an initial segment. For $t \in T$ let $\alpha(t) = \text{Min}\{\alpha : \text{for some } f \in \bigcup_{s \geq t} \mathcal{F}_s \text{ and } (I_f, J_f) \in \mathcal{Y} \text{ (as in 4.5 clause (e)')} \text{ we have } \{i : i \in \text{Dom}(f)$

and $f(i) \in N_\alpha\} \in J^+\}$.

Let $S = \{\beta < \lambda : \text{for some } t \in T \text{ we have } \beta = \alpha(t)\} \subseteq \lambda$. For each $\beta \in S$ choose $t_\beta \in T$ and $s_\beta, t_\beta \leq_T s_\beta$ such that $\beta = \alpha(t_\beta)$ and $f_\beta \in \mathcal{F}_{s_\beta}$ witness this. Let $\mathcal{U}_\beta = \text{Dom}(f_\beta)$ and let $(I_\beta, J_\beta) = (I_{f_\beta}, J_{f_\beta})$. Now without loss of generality $f_\beta \in N_{\beta+1}$

(hence $s_\beta, I_\beta, J_\beta \in N_{\beta+1}$ (as all the requirements on f_β have parameters in $N_{\beta+1}$)). First assume toward contradiction that S is stationary. Now as $\mathcal{Y} \in N_0, |\mathcal{Y}| < \lambda$ clearly $\mathcal{Y} \subseteq N_0$ hence for some $y \in \mathcal{Y}$ the set $S_y = \{\beta \in S : (I_\beta, J_\beta) = y\}$ is stationary.

Let $y = (I^*, J^*)$ and $S'_y = \{\beta \in S_y : N_\beta \cap \lambda = \beta\}$, clearly it is stationary. It suffices to show that $\langle f_\delta : \delta \in S'_y \rangle$ exemplifies $\text{NFr}_2(\lambda, \mathcal{Y})$ contradicting assumption (c)' from 4.5. If not, for some $\delta_1 < \delta_2$ in S_β we have $B = \{i : i \in \text{Dom}(f_{\delta_1}), i \in \text{Dom}(f_{\delta_2}) \text{ and } f_{\delta_1}(i) = f_{\delta_2}(i)\} \in I^+$, hence $t_{\delta_2} \in T_{\langle f_{\delta_1} \rangle}$ (see 4.5, clause (f)') hence

by an assumption there is t'_{δ_2} such that $t_{\delta_2} \leq_T t'_{\delta_2} \in T[f_{\delta_1}]$. But $\bar{\mathcal{F}}, f_{\delta_1}$ belong to $N_{\delta_1+1} \prec N_{\delta_2}$ hence $T_{\langle f_{\delta_1} \rangle} \in N_{\delta_1+1}$ but $T[f_{\delta_1}]$ has cardinality $\leq \mu$ (see clause (f)' of 4.5) hence $T_{\langle f_{\delta_1} \rangle} \subseteq N_{\delta_1+1}$ but $t'_{\delta_2} \in T[f_{\delta_1}]$ so $t'_{\delta_2} \in N_{\delta_1+1}$ hence $\mathcal{F}_{t_{\delta_2}} \in N_{\delta_1+1}$ hence (see 4.5, (d)') we have $F_{t_{\delta_2}} \subseteq N_{\delta_1+1}$ hence there is $f' \in \mathcal{F}_{t'_{\delta_2}} \cap N_{\delta_1+1}$ hence $\text{Rang}(f') \subseteq N_{\delta_1+1}$ contradicting the demand $\alpha(t_{\delta_2}) = \delta_2$. So S is not stationary.

Let E be a club of λ disjoint to S and we can find $\bar{N}' = \langle N'_\alpha : \alpha < \lambda \rangle$ like $\langle N_\alpha : \alpha < \lambda \rangle$ such that $E, \bar{N}' \in N_0$ so for $\bar{N}', S = \emptyset$. Now for each α we use the induction hypothesis on $X_\alpha = X \cap N'_{\alpha+1} \setminus N'_\alpha$ and $\langle \bar{\mathcal{F}}_t^{(\alpha)} : t \in T^{(\alpha)} \rangle$ where $T^{(\alpha)} = T \cap N'_{\alpha+1} \setminus N'_\alpha$ and $\mathcal{F}_t^{(\alpha)} = \{f \upharpoonright \mathcal{U} : \mathcal{U} \text{ is } \{i \in \text{Dom}(f) : f(i) \in X_\alpha\} \text{ and } f \in \mathcal{F}_t \cap N'_{\alpha+1}\}$.

Case 3: λ singular $> \mu$.

Let $\lambda = \sum_{i < \text{cf}(\lambda)} \lambda_i$, $\langle \lambda_i : i < \text{cf}(\lambda) \rangle$ increasingly continuous, $\lambda_0 > \mu^+$. We choose

by induction on $\zeta < \mu^+$, $\langle N_i^\zeta : i < \text{cf}(\lambda) \rangle$ such that:

- (a) N_i^ζ is \prec -increasing in i
- (b) $\langle \lambda_i : i < \text{cf}(\lambda) \rangle, X, \lambda, \mu, \bar{\mathcal{F}}$ all belong to N_0^ζ
- (c) $\lambda_i \subseteq N_i^\zeta$ and $\|N_i^\zeta\| = \lambda_i$
- (d) for each i , $\langle N_i^\zeta : \zeta \leq \mu^+ \rangle$ is \prec -increasingly continuous
- (e) $\langle \langle N_i^\varepsilon : i < \text{cf}(\lambda) \rangle : \varepsilon \leq \zeta \rangle \in N_i^{\zeta+1}$.

For each $i < \lambda$ and $\zeta < \mu^+$ and $(I, J) \in \mathcal{Y}$ let $\mathcal{F}_{(I,J)}^{\zeta,i}$ be a maximal family of functions $f \in \{f \upharpoonright \mathcal{U} : \mathcal{U} \in J_f^+, f \in \bigcup_{t \in T} \mathcal{F}_t, \mathcal{U} \subseteq \text{Dom}(f)\}$, $\text{Rang}(f) \subseteq X \cap N_i^\zeta$ and $f \neq g \in \mathcal{F}_{(I,J)}^{\zeta,i} \Rightarrow \{i : i \in \text{Dom}(f), i \in \text{Dom}(g) \text{ and } f(i) \neq g(i)\} \in I$. Without loss of generality $\mathcal{F}_{(I,J)}^{\zeta,i} \in N_{i+1}^\zeta$ and by 4.2(4) and assumption 4.5 clause (c)' we know $|\mathcal{F}_{(I,J)}^{\zeta,i}| \leq \lambda_i$, so a list of it of length $\leq \lambda_i$ belongs to $N_i^{\zeta+1}$ hence $\mathcal{F}_{(I,J)}^{\zeta,i} \subseteq N_i^{\zeta+1}$. So if $t \in T$ and we define $\alpha(t)$ as in Case 2 for $\langle N_\alpha^{\mu^+} : \alpha \leq \text{cf}(\mu) \rangle$, we get that $\alpha(t)$ is necessarily nonlimit. $\square_{4.12}$

Proof of 4.5.

Just by 4.12 above and 4.6 (reading Definition 4.11).

Proof of Lemma 3.2.

Just by 4.12 above and 4.6.

Proof of 3.11. We use 4.4 above.

1) Without loss of generality let λ be the set of points of $X, \mu = \chi, I = \{A \subseteq {}^\omega 2 : \text{the closure of } A \text{ is countable}\}, J$ the following ideal on ${}^\omega 2$

$$\{\mathcal{U} \subseteq {}^\omega 2 : |\mathcal{U}| < 2^{\aleph_0}\}$$

and

$$\mathcal{Y} = \{(I, J)\}.$$

So the conclusion (*) of 3.11 just means “for some regular $\kappa \in (\mu, \lambda]$ we have $\text{NFr}_1(\kappa, \mathcal{Y})$ ” and toward contradiction assume it fails. Clearly $\chi \geq \mu$. Also without loss of generality the set of points of X is λ , let $\text{cl} : [\lambda]^{\leq \mu} \rightarrow [\lambda]^{\leq \mu}$ be

$$\begin{aligned} \text{cl}(A) = \{ & \alpha : \alpha \in A \text{ or for some countable } B \subseteq A, \alpha \text{ belongs} \\ & \text{to the closure of } B \text{ in the topological space } X \text{ and} \\ & \text{cl}(B) \text{ has cardinality } \leq 2^{\aleph_0} \}. \end{aligned}$$

Let us consider the assumptions of 4.4. Now clause (a) is by the explicit choice of \mathcal{Y} above, also (b). Clause (c) is the assumption toward contradiction above, clause (d) (on cl) holds as clearly $A \in [\lambda]^{\leq \mu}$ implies $\text{cl}(A) = \cup \{\text{cl}(B) : B \in [A]^{\leq \aleph_0}\}$ and $[A]^{\leq \aleph_0}$ has cardinality $\leq \mu^{\aleph_0} = \chi^{\aleph_0} = \chi$ and for each countable B , contribute at most 2^{\aleph_0} points. Clause (e) holds by the properties of closure. Lastly, for clause (f) we define

$$\begin{aligned} \mathcal{P} = \{ & A : A \subseteq \lambda \text{ is a closed subset of } A, \\ & \text{has cardinality continuum and } X \upharpoonright A \text{ is homeomorphic to } {}^\omega 2 \}. \end{aligned}$$

So as all the assumptions of 4.4 holds so we can apply 4.6. There for $\theta = 2^{\aleph_0}$, if we can apply 4.6(1) we get $X \not\rightarrow ({}^\omega 2)_{2^{\aleph_0}}^1$. But $(\alpha)_\theta$ of 4.6 is immediate for any closed subspace Y of X homeomorphic to ${}^\omega 2$, we have $|Y \cap \bigcup_{\beta < \alpha} M_\beta| < 2^{\aleph_0}$ and for some

$A \in M_\alpha, |A| = 2^{\aleph_0}, Y \cap \text{cl}(A)$ contains a set Y' homeomorphic to ${}^\omega 2$, and this has 2^{\aleph_0} pairwise disjoint subspaces which $\in M_\alpha$ so at least one is disjoint to $\bigcup_{\beta < \alpha} M_\beta$ so

we are done by the choice of h_α .]

2) Similar, we just should be more accurate about closure; note that the topological closure of a countable set may have cardinality bigger than 2^{\aleph_0} . For $A \subseteq X$ let $\text{cl}(A) = \text{cl}(A, X) = \cup \{\text{Rang}(f) : f \text{ a one to one mapping from } \mathbb{R} \text{ to } X \text{ which is a homeomorphism and such that } Y_f = \{x \in \mathbb{R} : f(x) \in A\} \text{ is dense}\}$. But for any such f_1, f_2 , if some $Y \subseteq Y_{f_1} \cap Y_{f_2}$ is countable dense and $[x \in Y \Rightarrow f_1(y) = f_2(y)]$ then $f_1 = f_2$, so the proof is similar. $\square_{3.11}$

4.13 Concluding Remark. 1) Of course, we may replace in 3.2 the space ${}^\omega 2$ by many others, e.g. \mathbb{R} , or any Hausdorff Y^* with 2^{\aleph_0} points such that for any uncountable $A \subseteq Y^*$, for some countable $B \subseteq A, |\text{cl}_{Y^*}(B)| = 2^{\aleph_0}$ moreover if $Z \subseteq Y^*, |Z| < 2^{\aleph_0}$ for some uncountable $B' \subseteq \text{cl}_{Y^*}(B)$ we have $\text{cl}_{Y^*}(B')$ is disjoint to Z .

We can also add variants with \rightarrow_w replacing \rightarrow . As long as the space has $\leq 2^{\aleph_0}$ points, the only place we should be concerned is the proof of 3.11, we reconsider the choice of cl in the proof. In all cases for an embedding f from $Y \subseteq Y^*$ to X , let $\text{cl}(\text{Rang}(f)) = \{x \in X : \text{for some } y \in Y^*, f \cup \{\langle y, x \rangle\} \text{ is an embedding of}$

$Y^* \restriction (\mathcal{U} \cup \{x\})$ to $X \restriction ((\text{Rang}(f)) \cup \{y\})$ and $f^+ = f \cup \{\langle y, x \rangle : x, y \text{ as above}\}$. The point is that for this choice of $c\ell$, if $Y_1 \subseteq Y_2 \subseteq Y^*$, $Y_2 \subseteq c\ell_{Y^*}(X_1)$ if f embeds Y_2 into X with $\text{Rang}(f)$ not necessarily close, then $(f \restriction X_1)^+$ is a function from some $Y_3 \subseteq Y^*$ into X extending f .

2) We may like to add to 3.2 the case with continuum many colours that is let $(B_m)_{<\mu}^{+[\omega 2]}$ and $(B_m)_{<\mu}^+$ be defined like $(B)[\omega 2], (B)^+$, replacing $_{<cf(2^{\aleph_0})}^1$ by $_{<\mu}^1$ and we add $(B_m)_{<\aleph_2^+}^{+[\omega 2]}, (B)_{<\aleph_2^+}^+$ to the list of equivalent statements. Similarly for (A). More is proved $X \rightarrow (\omega 2)_{<\lambda}^1$ where X has λ points (or we get λ when we ask for compact X). The main point is adopting 1.2 (and 1.5).

For this we add also $(C_m)_{\aleph_2, \aleph_2, \aleph_2}$ where for $\kappa \geq \theta \geq \sigma$ we let

$(C_m)_{\kappa, \theta, \sigma}$ there are λ, S, f such that

- (a) $S \subseteq \lambda$ is stationary $> \kappa^+, \kappa > \theta \geq \sigma$
- (b) $\bar{f} = \langle f_\delta : \delta \in S \rangle$
- (c) $\text{Dom}(f_\delta) = \theta$, each $f_\delta(i)$ is a subset of $\delta \setminus i$ of cardinality $\leq \kappa$ and $\langle \min(f_\delta(i)) : i < \theta \rangle$ is increasing with limit δ (can ask $i < j < \theta \Rightarrow f(\sup(f_\delta(i))) < \min(f_\delta(j))$)
- (d) if $\delta_1 < \delta_2$ are in S then $\{i < \theta : f_{\delta_2}(i) \cap \bigcup_{j < \theta} f_{\delta_1}(j) \neq \emptyset\}$ has cardinality $< \sigma$
- (e) if $F_\ell : \lambda \rightarrow [\lambda]^{\leq \kappa}$ for $\ell = 0, 1$ and $F_0(\alpha) \in [\lambda \setminus \alpha]^{\leq \kappa}$, then for some $\delta \in S$ we have:
 - (α) f_δ is (F_0, F_1) -free which means:
for $i \neq j < \theta$, the set $F_1(f_\delta(i))$ is disjoint to $F_0(f_\delta(j))$
 - (β) there are $\langle \alpha_i : i < \theta \rangle$ such that $f_\delta(i) = F_0(\alpha_i)$ and $\sup[\bigcup_{j < i} f_\delta(j)] < \alpha_i$.

Similarly for (D). Why is this O.K.? See below, noting that we get more.

3) As before, $(B_m)^+ \Rightarrow (B_m)[\omega 2] \Rightarrow (A_m)[\omega 2]$ and $(B_m)^+ \Rightarrow (A_m)^+ \Rightarrow (A_m)[\omega 2]$, also easily $(C) \Rightarrow (C)_{\aleph_2, \aleph_2, \aleph_2}^+ : (B_m)^+ \Rightarrow (B)^+, (A_m)^+ \Rightarrow (A)^+, (B_m)[\omega 2] \Rightarrow (B)[\omega 2]$ and $(A_m)[\omega 2] \Rightarrow (A)[\omega 2]$

(f) if (F_0, F_1) is a pair of functions with domain λ and $F_0(i) \in [\lambda \setminus i]^{\leq \kappa}$

3A) The forcing in 2.6, with the role of A_ζ being replaced by $\bigcup_{i < \theta} f_\zeta(i)$ and $A_\zeta^p \subseteq$

$\bigcup_{i < \theta} f_\delta(i)$ such that $i < \theta \Rightarrow |A_\zeta^p \cap f_\delta(i)| \leq 1$ works.

4) Also

$\boxtimes_4 (D)_{\aleph_2, \aleph_2, \aleph_2}$ implies the consistency of $(B_m)_{<\aleph_2^+}^+$.

As before without loss of generality for some $\kappa = \kappa^{<\kappa} \geq \theta = 2^{\aleph_0}, \sigma$ are such that $(C)_{\kappa, \theta, \sigma}$ hold. Now we just need to repeat the proof of 1.2. The asymmetry in clause (d) does not hurt as if $\delta_2 \neq \delta_2, A_{\delta_1}^{p_1}, A_{\delta_2}^{p_2}$ are well defined, then it follows that $|A_{\delta_1}^{p_1} \cap A_{\delta_2}^{p_2}| < \sigma$.

In the crucial point we let $p^* \Vdash \text{"}\dot{c} : \lambda \rightarrow \mu \text{ for some } \mu < \lambda\text{"}$. Really less is enough: let $p^* \Vdash \text{"}\dot{Z} \subseteq \lambda \text{ is unbounded"}$ and we shall find q and $\delta \in S$ such that $p^* \leq q \in P$ and $q \Vdash \text{"}\dot{X}^* \restriction A_\delta^p \text{ is a copy of the space } Y \text{ (e.g. } \omega 2) \text{ and } A_\delta^p \subseteq Y\text{"}$. How? We define

$$F_0(\alpha) = \{\beta : \beta \in [\alpha, \lambda) \text{ and } p^* \Vdash \beta \neq \text{Min}(\dot{Z} \setminus \alpha)\}.$$

$F_1(\alpha) = \cup \{u^{p_{\alpha,i}} : i < \kappa\}$ where $\langle p_{\alpha,i} : i < \kappa \rangle$ is a maximal antichain above p^* such that $p_{\alpha,i}$ forces $\alpha \in \dot{Z}$ or forces $\alpha \notin \dot{Z}$.

Now we repeat the proof of 1.2, but instead deciding the colour we decide the right member of \dot{Z} .

5) Lastly, we get $(C)_{\kappa,\theta,\nu}^+$ from $(C)_{\kappa,\theta,\nu}$. So assume $\lambda > \kappa^+, \kappa > \theta \geq \sigma$ and $\langle A_\delta : \delta \in S \rangle$ are as in (C) and as before (by forcing) without loss of generality \diamond_S . Now we can actually prove $(C)_{\kappa,\theta,\sigma}$ for λ . So we prove

\boxtimes_5 if

- (α) $\lambda > \kappa^+, \kappa > \theta \geq \sigma, \kappa^\sigma < \lambda$
- (β) J an ideal on θ such that $(\forall A \in J^+)(\exists a \in J^+)(a \subseteq A)$
- (γ) $S \subseteq \lambda$ is stationary, $\bar{f} = \langle f_\delta : \delta \in S \rangle$, $f_\delta : \theta \rightarrow \theta$ increasing, $\delta_1 < \delta_2 \Rightarrow \{i < \theta : f_{\delta_1}(i) = f_{\delta_2}(i)\} \in J^+$
- (δ) \diamond_S .

Then $(C)_{\kappa,\theta,\sigma}$ as witnessed by λ .

So let $\langle (F_0^\delta, F_1^\delta) : \delta \in S \rangle$ be such that $F_\ell^\delta : \delta \rightarrow [\delta]^{<\kappa}$ for $\ell = 0, 1$ be such that: if $F_\ell : \lambda \rightarrow [\lambda]^{<\kappa}$ for $\ell = 0, 1$ then $S_{(F_0, F_1)} = \{\delta \in S : F_0 \restriction \delta = F_0^\delta \text{ and } F_1 \restriction \delta = F_1^\delta\}$ is stationary. We now choose by induction on $\delta \in S$ a function f_δ such that:

- (a) if there is a function f with domain θ satisfying the conditions below then f_δ is such a function, otherwise f_δ is constantly \emptyset
 - (α) $f(i) \in [\delta]^{<\kappa} \setminus \{\emptyset\}$
 - (β) $i < j \Rightarrow \sup(f_\delta(i)) < \min(f_\delta(j))$
 - (γ) for each $i < \theta$ for some $\alpha_i < \delta$ we have $F_0^\delta(\alpha_i) = f_\delta(i)$ and $\sup(\bigcup_{j < i} f(j)) < \alpha_i \leq \min f(i)$
 - (δ) $\langle \min(f(i)) : i < \theta \rangle$ converge to δ
 - (ε) for $i \neq j < \theta$ the set $F_1^\delta(f(i))$ and $F_6^\delta(f(j))$ are disjoint
 - (ζ) if $\delta_1 \in \delta \cap S$ then $\{i < \delta : f(i) \cap \bigcup_{j < \theta} f_{\delta_1}(j) \neq \emptyset\}$ has cardinality $< \sigma$.

Let $S^- = \{\delta \in S : f_\delta \text{ is not constantly } \emptyset\}$ and we suffice to prove that $\bar{f} = \langle f_\delta : \delta \in S^- \rangle$ is as required. Most clauses hold by the definition and we should check clause (e), so let F_0, F_1 be as there. Let $S_{F_0, F_1} = \{\delta \in S : F_0 \restriction \delta = F_0^\delta \text{ and } F_1 \restriction \delta = F_1^\delta\}$, so this set is stationary.

For every $\alpha \in S^* = \{\delta < \lambda : \text{cf}(\delta) = \kappa^+\}$ let $g(\alpha) = \sup(\alpha \cap F_1(\alpha)) < \alpha$ so g is constantly $\alpha(*)$ on some stationary $S^{**} \subseteq S$.

$E_0 = \{\delta < \lambda : \text{otp}(S^{**} \cap \delta) = \delta \text{ and } \alpha < \delta \Rightarrow \sup(F_0(\alpha)) < \delta \text{ and } \alpha < \delta \Rightarrow \sup(F_1(\alpha)) < \delta\}$.

Let $E_1^* = \{\delta < \lambda : \text{otp}(E_0 \cap \delta) = \delta\}$ and for $\delta \in E_1 \cap S_{F_0, F_1}$ let $A'_\delta = \{\alpha \in E_0 : \text{otp}(\alpha \cap E_0) \in A_\delta\}$, so $A_\delta \subseteq \delta = \sup(A_\delta)$, $\text{otp}(A_\delta) = \theta$ and $\delta_1 \neq \delta_2 \in E_1 \cap S_{F_0, F_1} \Rightarrow |A_{\delta_1} \cap A_{\delta_2}| < \sigma$.

Let $A_\delta = \{\alpha'_{\delta, i} : i < \theta\}$ increasingly and let $\alpha_{\delta, i} = \text{Min}(S^{**} \setminus (\alpha'_{\delta, i} + 1))$ so $\alpha_{\delta, i} < \alpha''_{\delta, i+1}$ (even $\alpha_{\delta, i} < \text{Min}(E_1 \setminus (\alpha'_{\delta, i} + 1))$ and choose f'_δ a function with domain θ by

$$f'_\delta(i) = F_0(\alpha_{\delta, i}) = F_0^\delta(\alpha'_i)$$

(the last equality as $F_\ell \upharpoonright \delta = F_\ell^\delta$ as $\delta \in S_{F_0, F_1}$).

Clearly $f'_\delta(i) = F_0(\alpha_i) \subseteq \text{Min}(E_1 \setminus (\alpha'_\ell + 1))$ and

$$\gamma \in f'_\delta(i) \Rightarrow F(\gamma) \subseteq \text{Min}(E_1 \setminus (\alpha'_{\delta, i} + 1)) \leq \alpha'_{\delta, i+1} < \alpha'_{\delta, i+1}$$

$$\gamma \in f'_\delta(i) \Rightarrow F(\gamma) \cap \alpha_{\delta, i} \subseteq \alpha(*) < \alpha_0$$

Now f'_δ satisfies almost all the requirements on f_δ and if $f'_\delta = f_\delta$ for stationarily many $\delta \in E_1 \cap S_{F_0, F_1}$ we are done. Let $W = \{\delta \in E_1 \cap S_{F_0, F_1} : f'_\delta \neq f_\delta\}$, we shall prove that W is not stationary - this is more than enough.

So for $\delta \in W$ necessarily for some $h(\delta) \in \delta \cap S$ we have

$$w_\delta = \{i < \theta : f'_\delta(i) \cap \bigcup_{j < \theta} f_{(\delta)}(j) \neq \emptyset\}$$

has cardinality $\geq \sigma$, so by Fodor's lemma for some $\delta(*)$ we have $W_1 = \{\delta \in W : h(\delta) = \delta(*)\}$ is stationary.

Similarly as $\theta^\sigma < \lambda = \text{cf}(\lambda)$ for some $w^* \in [\theta]^\sigma$, $w_2 = \{\delta \in w_1 : w^* \subseteq w_\delta\}$ is stationary. As $^\sigma[\bigcup_{j < \theta} f_{\delta(*)}(j)]^\sigma$ has cardinality κ^σ which is $< \lambda$ without loss of generality for

some $h^* : w^* \rightarrow \bigcup_{j < \delta} f_{\delta(*)}(j)$ the set

$$W_3 = \{\delta \in W_2 : (\forall i \in w^*)(h^*(i) \in f'_\delta(i) \cap \bigcup_{j < \theta} f_{\delta(*)}(j))\}$$

is stationary. So if $\delta_1 < \delta_2$ are in w_3 the set $\{i < \theta : f'_{\delta_1}(i) = f'_{\delta_2}(i)\}$ include w^* . But $f'_{\delta_1}(i) = f'_{\delta_2}(i)$ implies that $\alpha_i^{\delta_1} = \alpha_i^{\delta_2}$, hence $A_{\delta_1} \cap A_{\delta_2}$ has cardinality $\geq \sigma$ continuously.

6) W has a \boxtimes clause (δ) , we add: $\text{Rang}(f_\delta)$ is bound in δ ?

This is equivalent to: for some fixed $\mu < \lambda$, $(\forall \delta)(\text{Rang}(f_\delta) \subseteq \mu)$. Repeating the proof and replacing club of $C \in [\mu]^\mu$ we get clause $(C)_{\kappa, \theta, \sigma}$ witnessing λ with $\text{Rang}(f_\delta) \subseteq \mu$. We then get versions of the (A) 's and (B) 's with μ points.

(Note one special point: we should rephrase the "weak Δ -system argument, by using it on a tree with two levels.

7) Note that by part (5) we get a stronger version of the topological statements: for any λ (or μ in (6)) points there is a close copy of ${}^\omega 2$ (or the space Y) included in it. Of course, if we like the space to be compact this refers only to any set of λ (or μ)

points among the original ones. Note the Boolean Algebra of clopen sets (when Y has such a basis) satisfies the c.c.c. (remember in the cases only $u_{\zeta,2i}^p \cap u_{\zeta,2i+1}^p = \emptyset$ is demanded, the Boolean Algebra is free) so we cannot control the set of ultrafilters (= points), but if we allow more disjointness demand we may, but we have not considered it.

4.14 Claim. *Assume $\mu = \mu^{<\mu}$. There is a μ -complete μ^+ -c.c. forcing notion Q such that*

- \Vdash_Q “there is a function $h : {}^\mu\mu \rightarrow \mu$ such that
- (α) if $C \in V$ is a closed subset of ${}^\mu\mu$ of cardinality $\leq \mu$
then $\alpha < \mu \Rightarrow |C \cap h^{-1}\{\alpha\}| < \mu$
 - (β) if $A \in V$ is a subset of ${}^\mu\mu$ of cardinality $> \mu$
then $\alpha < \mu \Rightarrow |A \cap h^{-1}\{\alpha\}| = |A|$ ”.

Proof. As in the proof of 3.12, it suffices to prove:

- (*) assume $i^*, j^* < \mu$ and $\eta_{\alpha,i} \in {}^\mu\mu$ for $\alpha < \mu^+, i < i^*$ is with no repetitions and $C_{\alpha,j} \subseteq {}^\mu\mu$ is closed with $\leq \mu$ points for $\alpha < \mu^+, j < j^*$. Find $\alpha < \beta$ such that $i < i^*$ & $j < j^* \Rightarrow \eta_{\alpha,i} \notin C_{\beta,j}$.

Why (*) holds? Assume not. First choose $\delta^* < \mu^+$ such that:

- (**) if $\beta < \mu^+$ and $\zeta < \mu$ then for some $\alpha < \delta^*$ we have $i < i^* \Rightarrow \eta_{\alpha,i} \restriction \zeta = \eta_{\beta,i} \restriction \zeta$.

We can find β such that $\delta^* < \beta < \mu^+$ and $\{\eta_{\beta,i} : i < i^*\}$ is disjoint to $\bigcup_{j < j^*} C_{\delta^*,j}$.

β exists as $|\bigcup_{j < j^*} C_{\delta^*,j}| \leq \mu$. Let $\zeta^* < \mu$ be large enough such that $i < i^*$ & $j < j^* \Rightarrow \neg(\exists \nu)(\eta_{\beta,i} \restriction \zeta \triangleleft \nu \in C_{\delta^*,j})$. Lastly, choose $\alpha < \delta^*$ such that $i < i^* \Rightarrow \eta_{\alpha,i} \restriction \zeta = \eta_{\beta,i} \restriction \zeta$. Now $\langle \alpha, \delta^* \rangle$ can serve as (α, β) above. $\square_{4.14}$

APPENDIX: SIMILAR PROOFS

5.1 Proof of 1.5

Proof. We write the proof for part (1) and indicate the changes for part (2). Without loss of generality

$$\otimes_1 (\forall \alpha < \beta < \lambda)(\forall B \in [\lambda]^{<\lambda})(\exists^{\kappa^+} A \in \mathcal{A})[\{\alpha, \beta\} \subseteq A \ \& \ A \cap B \subseteq \{\alpha, \beta\}].$$

[Why? As we can use $\{\{2\alpha : \alpha \in A\} : A \in \mathcal{A}\}$, without loss of generality $\bigcup\{A : A \in \mathcal{A}\} = \{2\alpha : \alpha < \lambda\}$ and choose $A_{\alpha,\beta,\gamma} \in [\lambda]^\theta$ for $\alpha < \beta < \gamma < \lambda$ such that $\{\alpha, \beta\} \subseteq A_{\alpha,\beta,\gamma}$ and $\langle A_{\alpha,\beta,\gamma} \setminus \{\alpha, \beta\} : \alpha < \beta < \gamma < \lambda \rangle$ are pairwise disjoint subsets of $\{2\alpha + 1 : \alpha < \lambda\}$ each of cardinality θ and replace \mathcal{A} by $\mathcal{A}^* =: \mathcal{A} \cup \{A_{\alpha,\beta,\gamma} : \alpha < \beta < \gamma < \lambda\}$. Now clause (A), (D), (E) are not affected. Clearly clause $(B)_1$ holds (i.e. $\mathcal{A}^* \subseteq [\lambda]^\theta$ and $A \neq B \in \mathcal{A}^* \Rightarrow |A \cap B| < \sigma$). Also clause (C) is inherited by any extension of the original \mathcal{A} . Lastly for clause $(B)_2$, if $\mathcal{A}' \subseteq \mathcal{A}^*$, $|\mathcal{A}'| < \kappa$, let $\langle A_\zeta : \zeta < \zeta^* \rangle$ be a list of $\mathcal{A}' \cap \mathcal{A}$ as guaranteed by $(B)_2$ and let $\langle A_\zeta : \zeta \in [\zeta^*, \zeta^* + |\mathcal{A}' \setminus \mathcal{A}|] \rangle$ list with no repetitions $\mathcal{A}' \setminus \mathcal{A}$, now check.]

$$\otimes_2 \mathcal{B} \text{ is a basis of } Y^* \text{ of cardinality } \theta^*, \text{ and for part (2), } \mathcal{B} \text{ is as there.}$$

[Why? Straight.]

Let $\mathcal{A} = \{A_\zeta : \zeta < \lambda^*\}$ and $\mathcal{B} = \{b_i : i < \theta^*\}$.

We define a forcing notion P :

$p \in P$ has the form $p = (u, u_*, v, v_*, \bar{w}) = (u^p, u_*^p, v^p, v_*^p, \bar{w}^p)$ such that:

- (α) $u_* \subseteq u \in [\lambda]^{<\kappa}$
- (β) $v_* \subseteq v \in [\lambda^*]^{<\kappa}$
- (γ) $\bar{w} = \bar{w}^p = \langle w_{\zeta,i} : \zeta \in v_* \text{ and } i < \theta^* \rangle = \langle w_{\zeta,i}^p : \zeta \in v_*, i < \theta^* \rangle$
- (δ) $w_{\zeta,i} \subseteq u_*$ and $b_i \cap b_j = \emptyset \Rightarrow w_{\zeta,i} \cap w_{\zeta,j} = \emptyset$ this is toward being Hausdorff
- (ε) $\zeta \in v_* \Rightarrow A_\zeta \subseteq u$
- (ζ) letting $A_\zeta^p = \cup\{w_{\zeta,i} : i < \theta^*\} \cap A_\zeta$ for $\zeta \in v_*^p$ it has cardinality θ and for simplicity even order type θ and letting $\langle \gamma_{\zeta,j}^p : j < \theta \rangle$ list its members with no repetitions we have $w_{\zeta,i}^p \cap A_\zeta^p = \{\gamma_{\zeta,j}^p : j < \theta \text{ and } j \in b_i\}$
- (η) if $\zeta \in v_*^p, i < \theta^*$ and $\xi \in v_*^p$ then the set $\mathcal{U}_{\zeta,\xi,i}^p$ is an open subset (for part (2), clopen subset) of the space Y^* where $\mathcal{U}_{\zeta,\xi,i}^p =: \{j < \theta : \gamma_{\xi,j}^p \in w_{\zeta,i}^p\}$.

$$\oplus \text{ convention if } \zeta \in \lambda^* \setminus v_*^p \text{ we stipulate } w_{\zeta,i}^p = \emptyset.$$

The order is: $p \leq q$ iff $u^p \subseteq u^q, u_*^p = u_*^q \cap u^p, v^p \subseteq v^q, v_*^p = v_*^q \cap v^p$ and $\zeta \in v_*^p \Rightarrow w_{\zeta,i}^p = w_{\zeta,i}^q \cap u^p$.

Clearly

$$(*)_0 \ P \text{ is a partial order.}$$

What is the desired space in V^P ? We define a P -name \dot{X}^* as follows:

set of points $\bigcup \{u_*^p : p \in G_P\}$

The topology is defined by the following basis:

$\{\bigcap_{\ell < n} \mathcal{U}_{\zeta_\ell, i_\ell} : n < \omega, \zeta_\ell < \lambda^*, i_\ell < \theta^*\}$ where

$\mathcal{U}_{\zeta, i}^p[G_P] = \bigcup \{w_{\zeta, i}^p : p \in G_P, \zeta \in v_*^p\}$

(for part (2), also their compliments and even their Boolean combinations)

(*)₁ for $\alpha < \lambda$ and $p \in P$ will have $p \Vdash \text{"}\alpha \in \dot{X}' \text{"}$ iff $\alpha \in u_*^p$ and $p \Vdash \text{"}\alpha \notin \dot{X}' \text{"}$ iff $\alpha \in u_\alpha^p \setminus u_*^p$

(*)₂ P is κ -complete, in fact if $\langle p_\varepsilon : \varepsilon < \delta \rangle$ is increasing in P and $\delta < \kappa$ then $p = \bigcup_{\varepsilon < \delta} p_\varepsilon$ is an upper bound where $u^p = \bigcup_{\varepsilon < \delta} u^{p_\varepsilon}$, $u_*^p = \bigcup_{\varepsilon < \delta} u_*^{p_\varepsilon}$, $v^p = \bigcup_{\varepsilon < \delta} v^{p_\varepsilon}$, $v_*^p = \bigcup_{\varepsilon < \delta} v_*^{p_\varepsilon}$ and $w_{\zeta, i}^p = \bigcup_{\varepsilon < \delta} w_{\zeta, i}^{p_\varepsilon}$ and $w_{\zeta, i}^p = \bigcup \{w_{\zeta, i}^{p_\varepsilon} : \zeta \in v_*^{p_\varepsilon}, \varepsilon < \delta\}$
[why? straight]

(*)₃ $P' = \{p \in P : \text{if } \zeta < \lambda^* \text{ and } |A_\zeta \cap u^p| \geq \sigma \text{ then } \zeta \in v^p\}$ is a dense subset of P
[why? for any $p \in P$ we define by induction on $\varepsilon \leq \sigma^+$: $p_\varepsilon \in P$, increasingly continuous with ε . Let $p_0 = p$, if p_ε is defined, we define $p_{\varepsilon+1}$ by

$$v^{p_{\varepsilon+1}} = \{\zeta < \lambda^* : \zeta \in v^{p_\varepsilon} \text{ or } |A_\zeta \cap u^{p_\varepsilon}| \geq \sigma\}$$

$$v_*^{p_{\varepsilon+1}} = v_*^{p_\varepsilon}$$

$$u^{p_{\varepsilon+1}} = u^{p_\varepsilon} \cup \bigcup \{A_\zeta : \zeta \in v^{p_{\varepsilon+1}}\}$$

$$u_*^{p_{\varepsilon+1}} = u_*^{p_\varepsilon} (= u_*^p)$$

$$w_{\zeta, i}^{p_{\varepsilon+1}} \text{ is: } w_{\zeta, i}^{p_\varepsilon} \text{ if } \zeta \in v_*^{p_\varepsilon}, i < \theta^*$$

(and there are no other cases).

By assumption (A)(ii), the set $v^{p_{\varepsilon+1}}$ has cardinality $< \kappa$, so $p_{\varepsilon+1}$ belongs to P .

Clearly $p_\varepsilon \leq p_{\varepsilon+1} \in P$.

Now for ε limit let $p_\varepsilon = \bigcup_{\xi < \varepsilon} p_\xi$. So we can carry the definition. Now $p_{\sigma^+} = \bigcup_{\varepsilon < \sigma^+} p_\varepsilon$ is

as required because if $A_\zeta \in \mathcal{A}$, $|A_\zeta \cap u^{p_{\sigma^+}}| \geq \sigma$ then for some $\varepsilon < \sigma^+$, $|A_\zeta \cap u^{p_\varepsilon}| \geq \sigma$ hence $\zeta \in v^{p_{\varepsilon+1}}$ hence $A_\zeta \subseteq u^{p_{\varepsilon+1}} \subseteq u^{p_{\sigma^+}}$.

Note that we use here $\sigma^+ < \kappa$.]

(*)₄ P satisfies the κ^+ -c.c.

[Why? Also easy. Let $p_j \in P$ for $j < \kappa^+$, without loss of generality $p_j \in P'$ for $j < \kappa^+$. Now by the Δ -system lemma for some unbounded $S \subseteq \kappa^+$ and

$v^\otimes \in [\lambda^*]^{<\kappa}$, $u^\otimes \in [\lambda]^{<\kappa}$ we have:

$j \in S \Rightarrow v^\otimes \subseteq v^{p_j}$ & $u^\otimes \subseteq u^{p_j}$ and $\langle v^{p_j} \setminus v^\otimes : j \in S \rangle$ are pairwise disjoint and $\langle u^{p_j} \setminus u^\otimes : j \in S \rangle$ are pairwise disjoint. Without loss of generality $\text{otp}(v^{p_j})$, $\text{otp}(u^{p_j})$ are constant for $j \in S$ and any two p_i, p_j are isomorphic over v^\otimes, u^\otimes (if not clear see 1.5).

Now for $j_1, j_2 \in S$, p_{j_1}, p_{j_2} are compatible because of the following $(*)_5$

$(*)_5$ assume $p^1, p^2 \in P$ satisfies

- (i) $v_*^{p^1} \cap (v^{p^2} \setminus v_*^{p^2}) = \emptyset$ and $u_*^{p^1} \cap (u^{p^2} \setminus u_*^{p^2}) = \emptyset$
- (ii) $v_*^{p^2} \cap (v^{p^1} \setminus v_*^{p^1}) = \emptyset$ and $u_*^{p^2} \cap (u^{p^1} \setminus u_*^{p^1}) = \emptyset$
- (iii) if $\zeta \in v_*^{p^1} \cap v_*^{p^2}$ then $A_\zeta^{p^1} = A_\zeta^{p^2}$ and
 $i < \theta^* \Rightarrow w_{\zeta,i}^{p^1} \cap (u^{p^1} \cap u^{p^2}) = w_{\zeta,i}^{p^2} \cap (u^{p^1} \cap u^{p^2})$
- (iv)₁ if $\zeta \in v_*^{p^1} \setminus v_*^{p^2}$ then $|A_\zeta \cap u^{p^2}| < \sigma$ or just $|A_\zeta^{p^1} \cap u^{p^2}| < \sigma$
- (iv)₂ similarly³ for $\zeta \in v_*^{p^2} \setminus v_*^{p^1}$

then there is $q \in P$ such that:

- (a) $v^q = v^{p^1} \cup v^{p^2}$
- (b) $v_*^q = v_*^{p^1} \cup v_*^{p^2}$
- (c) $u^q = u^{p^1} \cup u^{p^2}$
- (d) $u_*^q = u_*^{p^1} \cup u_*^{p^2}$
- (e) $p^1 \leq q, p^2 \leq q$.

[Why? To define the condition q we just have to define $w_{\zeta,i}^q$ (for $\zeta \in v_*^q = v_*^{p^1} \cup v_*^{p^2}$ and $i < \theta^*$). If $\zeta \in v_*^{p^1} \cap v_*^{p^2}$ we let $w_{\zeta,i}^q = w_{\zeta,i}^{p^1} \cup w_{\zeta,i}^{p^2}$ for $i < \theta^*$.

Now for $\ell = 1, 2$, let $v_*^{p^\ell} \setminus v_*^{p^{3-\ell}}$ be listed as $\langle \Upsilon(\varepsilon, \ell) : \varepsilon < \varepsilon_\ell^* \rangle$ with no repetitions such that $B_\varepsilon^\ell =: A_{\Upsilon(\varepsilon, \ell)}^{p^\ell} \cap (\bigcup_{\xi < \varepsilon} A_{\Upsilon(\xi, \ell)}^{p^\ell} \cup u^{p^{3-\ell}})$ is of cardinality $< \sigma$.

[Why possible? By the assumption $(B)_2$ and clause (iv) above.]

Now for each $\zeta \in v_*^{p^{3-\ell}} \setminus v_*^{p^\ell}$ we choose by induction on $\varepsilon < \varepsilon_\ell^*$ the sequence $\langle w_{\zeta,i}^{\ell, \varepsilon} : i < \theta^* \rangle$ such that

- 1) $w_{\zeta,i}^{\ell, \varepsilon} \subseteq u^{p^{3-\ell}} \cup \bigcup_{\xi < \varepsilon} A_{\Upsilon(\xi, \ell)}^{p^\ell}$.
- 2) $w_{\zeta,i}^{\ell, \varepsilon}$ is increasingly continuous with ε .
- 3) $w_{\zeta,i}^{\ell, 0} = w_{\zeta,i}^{p^{3-\ell}}$.
- 4) $\varepsilon' < \varepsilon \Rightarrow w_{\zeta,i}^{\ell, \varepsilon} \cap (u^{p^{3-\ell}} \cup \bigcup_{\xi < \varepsilon_1} A_{\Upsilon(\xi, \ell)}^{p^\ell}) = w_{\zeta,i}^{\ell, \varepsilon'}$.
- 5) if $i < j < \theta^*$ and $b_i \cap b_j = \emptyset$ (hence $w_{\zeta,i}^{p^\ell} \cap w_{\zeta,j}^{p^\ell} = \emptyset$) then $w_{\zeta,i}^{\ell, \varepsilon} \cap w_{\zeta,j}^{\ell, \varepsilon} = \emptyset$.
- 6) $\{j < \theta : \gamma_{\Upsilon(\varepsilon, \ell), j}^{p^\ell} \in w_{\zeta,i}^{\ell, \varepsilon+1}\}$ is an open set in Y^* (for part (2): clopen).]

³note that if $p^1, p^2 \in P'$, then clauses $(iv)_1, (iv)_2$ holds automatically.

For $\varepsilon = 0$ use clause (3) and for limit ε take unions (see clause (2)). Suppose we have defined for ε and let us define for $\varepsilon + 1$. By an assumption above B_ε^ℓ has cardinality $< \sigma$ and so $Z_\varepsilon^\ell = \{j < \theta : \gamma_{\Upsilon(\varepsilon, \ell), j}^{p_\ell} \in B_\varepsilon^\ell\}$ is a subset of θ of cardinality $< \sigma$. Hence, by assumption (E), we can find a sequence $\langle t_j(\varepsilon, \ell) : j \in Z_\varepsilon^\ell \rangle$ such that: $t_j(\varepsilon, \ell) < \theta^*$ and $j \in b_{t_j(\varepsilon, \ell)}$ for $j \in Z_\varepsilon^\ell$ and $\langle b_{t_j(\varepsilon, \ell)} : j \in Z_\varepsilon^\ell \rangle$ is a sequence of pairwise disjoint subsets of Y^* .

Lastly, we let

$$w_{\zeta, i}^{\ell, \varepsilon+1} = w_{\zeta, i}^{\ell, \varepsilon} \cup \left\{ \gamma_{\Upsilon(\varepsilon, \ell), s}^{p_\ell} : \text{for some } j \in Z_\varepsilon^\ell \text{ we have :} \right. \\ \left. \gamma_{\Upsilon(\varepsilon, \ell), t_j(\varepsilon, \ell)}^{p_\ell} \in w_{\zeta, i}^{\ell, \varepsilon} \text{ and } s \in b_{t_j(\varepsilon, \ell)} \right\}.$$

Clearly this is O.K. and we are done. Remember that the union of $< \sigma$ set from \mathcal{B} is clopen for part (2).]

(*)₆ in (*)₅ if in addition for $\ell = 1, 2$ we have $Z_\ell \subseteq u^{p_\ell} \setminus u^{p^{3-\ell}}$ such that $(\forall \zeta \in v_*^{p_\ell}) [|A_\zeta^{p_\ell} \cap Z_\ell| < \sigma]$ then we may add to the conclusion

$$\ell \in \{1, 2\}, \zeta \in v_*^{p^{3-\ell}} \setminus v_*^{p_\ell}, i < \theta^* \Rightarrow w_{\zeta, i}^q \cap Z_\ell = \emptyset.$$

More generally if $g_\ell : (v_*^{p^{3-\ell}} \setminus v_*^{p_\ell}) \times \theta^* \times Z_\ell \rightarrow \{0, 1\}$ we can add

$$\ell \in \{1, 2\}, \zeta \in v_*^{p^{3-\ell}} \setminus v_*^{p_\ell}, i < \theta^*, \gamma \in Z_\ell \Rightarrow [\gamma \in w_{\zeta, i}^q \leftrightarrow g_\ell(\zeta, i, \gamma) = 1].$$

[Why? When for $\zeta \in v_*^{p^{3-\ell}} \setminus v_*^{p_\ell}$, we define $\langle w_{\zeta, i}^{\ell, \varepsilon} : i < \theta^* \rangle$ by induction on ε we add

$$(7) \ i < \theta^*, \gamma \in Z_\ell \cap (u^{p^{3-\ell}} \cup \bigcup_{\xi < \varepsilon} A_{\Upsilon(\xi, \ell)}^{p_\ell}) \text{ implies } \gamma \in w_{\zeta, i}^{\ell, \varepsilon} \leftrightarrow g_\ell(\zeta, i, \gamma) =$$

1. In the proof when we use clause (E), instead of using $B_\varepsilon^\ell = A_{\zeta(\varepsilon, \ell)}^{p_\ell} \cap (\bigcup_{\xi < \varepsilon} A_{\zeta(\xi, \ell)}^{p_\ell} \cup u^{p^{3-\ell}})$ we use $B_\varepsilon^\ell = A_{\zeta(\varepsilon, \ell)}^{p_\ell} \cap (\bigcup_{\xi < \varepsilon} A_{\zeta(\xi, \ell)}^{p_\ell} \cup u^{p^{3-\ell}} \cup Z_\ell)$ which still has cardinality $< \sigma$.]

Now we come to the main point

(*)₇ in V^P , if $i(*) < \text{cf}(\theta)$ and $X^* = \bigcup_{i < i(*)} X_i$ then some closed $Y \subseteq X^*$ is

homeomorphic to Y^* .

[Why? Toward contradiction assume $p^* \in P$ and $p^* \Vdash_P \langle \dot{X}_i : i < i(*) \rangle$ is a counterexample to (*)₇].

Without loss of generality $p^* \Vdash_P \langle \dot{X}_i : i < i(*) \rangle$ is a partition of X^* , i.e. of λ .

For each $\alpha < \lambda$ let $\langle p_{\alpha, j}, i_{\alpha, j} : j < \kappa \rangle$ be such that:

- (i) $\langle p_{\alpha, j} : j < \kappa \rangle$ is a maximal antichain of P above p^*
- (ii) $p_{\alpha, j} \Vdash_P \text{“} \alpha \in \dot{X}_{i_{\alpha, j}} \text{”}$, so $i_{\alpha, j} < i(*)$
- (iii) $p^* \leq p_{\alpha, j}$.

Now choose a function F , $\text{Dom}(F) = \lambda$ as follows:

$$F(\alpha) \text{ is } \bigcup \{u^{p_{\alpha,j}} : j < \kappa\}.$$

So we can find $\zeta(*) < \lambda^*$ and $A \subseteq A_{\zeta(*)}$ of order type θ such that: if $\alpha \neq \beta$ are from A then $\alpha \notin F(\beta)$. Let $A = \{\beta_\varepsilon : \varepsilon < \theta\}$ with no repetitions. Now we shall choose by induction on $\varepsilon \leq \theta, p_\varepsilon, g_\varepsilon$ and if $\varepsilon < \theta$ also $j_\varepsilon < \kappa$ such that:

- (a) $p_\varepsilon \in P$ and $u^{p_\varepsilon} = u^{p^*} \cup \bigcup_{\varepsilon(1) < \varepsilon} u^{p_{\beta_\varepsilon(1), j_\varepsilon(1)}}$
 $v^{p_\varepsilon} = v^{p^*} \cup \bigcup_{\varepsilon(1) < \varepsilon} v^{p_{\beta_\varepsilon(1), j_\varepsilon(1)}}$
 $v_*^{p_\varepsilon} = v_*^{p^*} \cup \bigcup_{\varepsilon(1) < \varepsilon} v_*^{p_{\beta_\varepsilon(1), j_\varepsilon(1)}}$
 $w_{\zeta,i}^{p_\varepsilon} = w_{\zeta,i}^{p^*} \cup \bigcup_{\varepsilon(1) < \varepsilon} w^{p_{\beta_\varepsilon(1), j_\varepsilon(1)}} \text{ (remember the convention } \oplus)$
 (so $p_0 = p^*$)
- (b) $j_\varepsilon = \text{Min}\{j < \kappa : p_{\beta_\varepsilon,j} \text{ is compatible with } p_\varepsilon\}$
- (c) g_ε is a function, increasing with ε , from $v_*^{p_\varepsilon} \times \theta^*$ into the family of open subsets of Y^* (for part (2), clopen)
- (d) if $b_{i_1} \cap b_{i_2} = \emptyset$ then $g_\varepsilon(\zeta, i_1) \cap g_\varepsilon(\zeta, i_2) = \emptyset$ (if defined)
- (e) letting $\Upsilon_\varepsilon = \text{otp}\{\xi < \varepsilon : i_{\beta_\varepsilon, j_\varepsilon} = i_{\beta_\xi, j_\xi}\}$ we have for $\zeta \in v_*^{p_\varepsilon}$:

$$\beta_\varepsilon \in w_{\zeta,i}^{p_{\varepsilon+1}} \Leftrightarrow \Upsilon_\varepsilon \in g_\varepsilon(\zeta, i)$$

- (f) p_ε is increasing continuous.

No problem to carry the definition. As for ε successor, for this $(*)_6$ was prepared. In limit ε take union. In all cases j_ε is well defined by clause (i) above. Let $i^* < i(*)$ be minimal such that the set $Z = \{\varepsilon < \theta : i_{\beta_\varepsilon, j_\varepsilon} = i^*\}$ has cardinality θ . Note: $\zeta(*) \notin v^{p_{\beta_\varepsilon, j}}$ as $A \cap F(\beta_\varepsilon)$ is a singleton so $|A \cap u^{p_{\beta_\varepsilon, j}}| \leq 1$ and $p_{\beta_\varepsilon, j} \in P'$. Now we define p :

$$u^p = u^{p_\theta}$$

$$v^p = v^{p_\theta} \cup \{\zeta(*)\}$$

$$v_*^p = v_*^{p_\theta} \cup \{\zeta(*)\}$$

$w_{\zeta,i}^p$ is

- (α) $w_{\zeta,i}^{p_\theta}$ if $\zeta \in v^{p_\theta}$
- (β) $\{\beta_\varepsilon : \varepsilon \in Z \text{ and } \text{otp}(Z \cap \varepsilon) \in b_i\}$ if $\zeta = \zeta(*)$ so
- (γ) $A_{\zeta(*)}^p = \{\beta_\varepsilon : \varepsilon \in Z\}$ and $\gamma_{\zeta(*)}^p$ is the ε -th member of $A_{\zeta(*)}^p$.

We can easily check that $p \in P$ and $p^* \leq p_{\beta_\varepsilon, j_\varepsilon} \leq p \in P$ (but we do not ask $p_\varepsilon \leq p$). Clearly p forces that $\{\beta_\varepsilon : \varepsilon \in Z\}$ is included in one X_i .

Let $g : \theta \rightarrow \lambda$ be $g(\xi) = \beta_\varepsilon$ when $\xi < \theta, \varepsilon \in Z$, $\text{otp}(Z \cap \varepsilon) = \xi$. Now $p \geq p^*$ and we are done by $(*)_8$ below.]

$(*)_8$ if $p \in P$ and $\zeta \in v_*^p$ then

$p \Vdash$ “the mapping $j \mapsto \gamma_{\zeta, j}^p$ for $j < \theta$ is a homeomorphism from Y^* onto the closed subspace $\underline{X} \restriction \{\gamma_{\zeta, j}^p : j < \theta\}$ of \underline{X} ”

[Why? Let $p \in G, G \subseteq P$ is generic over V .

(α) If $b \in \mathcal{B}$, then for some open set \mathcal{U} of \underline{X} (clopen for part (2)) we have

$$\mathcal{U} \cap \{\gamma_{\zeta, j}^p : j < \theta\} = \{\gamma_{\zeta, j}^p : j \in b\}$$

[Why? As $b = b_i$ for some $i < i(*)$ and p forces that $w_{\zeta, i} \cap \{\gamma_{\zeta, j}^p : j < \theta\} = \{\gamma_{\zeta, j}^p : j \in b_i\}$.]

(β) If b is an open set for Y^* , then for some open subset \mathcal{U} of \underline{X} we have

$$\mathcal{U} \cap \{\gamma_{\zeta, j}^p : j < \theta\} = \{\gamma_{\zeta, j}^p : j \in b\}$$

[Why? As $b = \bigcup_{i \in Z} b_i$ for some $Z \subseteq \theta^*$ and apply clause (α)]

(γ) if \mathcal{U} is an open subset of \underline{X} and $\gamma_{\zeta, j(*)}^p \in \mathcal{U}$ (and $\zeta \in u_*^p$), then for some $i(*) < \theta^*$ we have

$$\gamma_{\zeta, j(*)}^p \in w_{\zeta, i(*)}^p \cap \{\gamma_{\zeta, j}^p : j < \theta\} \subseteq \mathcal{U}_{\zeta, i(*)} \cap \{\gamma_{\zeta, j}^p : j < \theta\} \subseteq \mathcal{U}.$$

[Why? By the definition of the topology \underline{X} we can find $n < \omega, \xi_\ell < \lambda^*$

and $i_\ell < \theta^*$ such that $\gamma_{\zeta, j(*)}^p \in \bigcap_{\ell < n} \mathcal{U}_{\xi_\ell, i_\ell}[G]$. We can find $q \in P$ such

that $p \leq q$ and $\xi_\ell \in v_*^q$ for $\ell < n$. For each $\ell < n$, by clause (η) in the definition of P we have $\mathcal{U}_{\zeta, \xi_\ell, j_\ell}^q$ is an open set for Y^* , and necessarily $j(*) \in \mathcal{U}_{\zeta, \xi_\ell, j_\ell}^q$. Let $i(*)$ be such that $j(*) \in b_{i(*)} \subseteq \bigcap_{\ell < n} \mathcal{U}_{\zeta, \xi_\ell, j_\ell}^q$ hence

$$\gamma_{\zeta, j(*)}^p \in \mathcal{U}_{\zeta, i(*)}[G] \cap \{\gamma_{\zeta, j}^p : j < \theta\} \subseteq \bigcap_{\ell < n} \mathcal{U}_{\xi_\ell, j_\ell}[G] \subseteq \mathcal{U} \text{ as required. So}$$

$i(*)$ is as required.]

(δ) $\{\gamma_{\zeta, j}^p : j < \theta\}$ is a closed subset of \underline{X}

[Why? Let $\beta \in \lambda \setminus \{\gamma_{\zeta, j}^p : j < \theta\}$ and let $p \leq q \in P$; it suffices to find $q^+, q \leq q^+ \in P$ and $\xi \in v_*^{q^+}$ and $i < \theta^*$ such that $\beta \in w_{\xi, i}^{q^+}$ and $w_{\xi, i}^{q^+} \cap \{\gamma_{\zeta, j}^p : j < \theta\} = \emptyset$. Without loss of generality $\beta \in u_*^q$.

We can find a set $u \subseteq u_*^q$ such that $\beta \in u, A_\zeta^q \cap u = \emptyset$ and $\zeta' \in v_*^q \Rightarrow \{j < \theta : \gamma_{\zeta', j}^q \in u\}$ is a clopen subset of Y , (just as in the proof of $(*)_5$). We can find by \otimes_1 $\xi \in \lambda^* \setminus v^q$ such that $\{\emptyset\} = A_\xi \cap u_*^q$ (why? apply \otimes_1 with $\alpha < \beta \in \lambda \setminus u^q$ and $B = u^q$) and let $\gamma_{\varepsilon, i} \in A_\xi$ for $i < \theta$ be increasing. We define q^+ as follows.

$$v^{q+} = v^q \cup \{\xi\}$$

$$v_*^{q+} = v_*^q \cup \{\xi\}$$

$$u^{q+} = u^q \cup A_\xi$$

$$u_*^{q+} = u_*^q \cup \{\gamma_{\xi,j} : j < \theta\}$$

$w_{\zeta,i}^{q+}$ is $w_{\zeta,i}^q$ if $\zeta \in v_*^q$ and is $\{\gamma_{\xi,j} : j \in b_i\} \cup u$ if $\zeta = \xi$ & $0 \in b_i$ and is $\{\gamma_{\xi,j} : j \in b_i\}$ if $\zeta = \xi$ & $0 \notin b_i$.

Lastly, we would like to know that X is a Hausdorff space. We prove more

(*)₉ In V^P if $u_1 \subseteq u_2 \in [\lambda]^{<\sigma}$ then for some ζ, i we have

$$w_{\zeta,i} \cap u_2 \cap X = u_1 \cap X$$

[Why? Let $p_0 \in P$ force $u_1 \subseteq u_2$ form a counterexample, as P is κ -complete some $p_1 \geq p_0$ forces $u_1 = u_2, u_2 = u_2$ and $p_1 \in P'$. Necessarily $u_2 \subseteq u_*^{p_1}$.

Let $\zeta(*) \in \lambda^* \setminus v^{p_1}$ be such that $A_{\zeta(*)} \cap u^{p_1} = \emptyset$ (as in the proof of (*)₈). Let $\gamma_{\zeta(*),j} \in A_{\zeta(*)}$, for $j < \theta$ be increasing. Let $u \subseteq u_*^{p_1}$ be such that $u \cap u_2 = u_1$ and $\zeta' \in v_*^{p_1} \Rightarrow \{j < \theta : \gamma_{\zeta',j}^{p_1} \in u\}$ is clopen in Y (exists as in the proof of (*)₅) and define $q \in P$:

$$u^q = u^{p_1} \cup u_2$$

$$u_*^q = u_*^{p_1} \cup (u_2 \setminus u^{p_1})$$

$$v^q = v^p \cup \{\zeta(*)\}$$

$$v_*^q = v_*^{p_1} \cup \{\zeta(*)\}$$

$w_{\zeta,i}^q$ is: $w_{\zeta,i}^{p_1}$ if $\zeta \in v^q$, is $\{\gamma_{\zeta(*),j} : j \in b_i\} \cup u$ if $\zeta = \zeta(*)$ & $0 \in b_i$ and is $\{\gamma_{\zeta(*),j} : j \in b_i\}$ if $\zeta = \zeta(*)$ & $0 \notin b_i$.]

Together all is done. □_{1.2}

5.2 Proof of 1.6(2)

Saharon: after 5.1.

5.3 Proof of 4.3

Proof. Let $F : \lambda \rightarrow [\lambda]^{\leq \kappa}$ be given. Choose by induction on $\zeta \leq \lambda$ a set $U_\zeta \subseteq \lambda$ and $g_\zeta : U_\zeta \rightarrow \kappa^+$, both increasingly continuous with ζ such that:

- (*) (i) if $\alpha \in U_\zeta$ then $F(\alpha) \subseteq U_\zeta$ and
- (ii) if $\alpha \in U_\zeta$ then $F(\alpha) \setminus \{\alpha\} \subseteq \{\beta \in U_\zeta : g_\zeta(\beta) \neq g_\zeta(\alpha)\}$.

For $\zeta = 0$ let $U_\zeta = \emptyset = g_\zeta$, for ζ limit take unions. If $U_\zeta = \lambda$, $U_{\zeta+1} = U_\zeta$, $g_{\zeta+1} = g_\zeta$, otherwise let $\alpha_\zeta = \text{Min}\{\lambda \setminus U_\zeta\}$ and let $W_\zeta \in [\lambda]^{\leq \kappa}$ be such that $\alpha_\zeta \in W_\zeta$ and $(\forall \alpha \in W_\zeta)[F(\alpha) \subseteq W_\zeta]$. Let $\varepsilon_\zeta = \sup\{g_\zeta(\beta) : \beta \in U_\zeta \cap W_\zeta\}$ so $\varepsilon_\zeta < \kappa^+$ and let $U_{\zeta+1} = U_\zeta \cup W_\zeta$, $g_{\zeta+1}$ extends g_ζ such that $g_{\zeta+1} \upharpoonright (W_\zeta \setminus U_\zeta)$ is one to one with range $[\varepsilon, \varepsilon + \kappa)$.

Now applying $(C)^+$ to the partition which $\bigcup_{\zeta} g_\zeta$ defines, we get some $A \in \mathcal{A}$ on

which $\bigcup_{\zeta} g_\zeta$ is constant so by $(*)(ii)$ we are done. $\square_{2.1}$

5.4 Proof of 4.13(2)

Saharon.

PRIVATE APPENDIX

moved from p.2

Problem: 1) Assume “there is a supercompact cardinal” (as in [Sh 108], [HJSh 249]) (for (A), (B), (C) or “there is a 2-huge cardinal” (as in [HJSh 249] for (B), (C)

- (A) can we get the consistency of the assumption of 2.2? (not just of 2.5)?
- (B) can we get the consistency required for 2.5 or even 2.6 for $X' \rightarrow (Y^*)_{\theta}^1, \theta > \kappa$?

For this it suffices:

- (*) if $h : \lambda \rightarrow \kappa^+$ then for some $A \in \mathcal{A}$ and $\zeta < \kappa^+$ we have $|A \cap h^{-1}\{\zeta\}| = \theta$.

2) Can we get the examples with GCH?

Remark. Seems easier to get $\theta > \kappa$, if say the space has size \aleph_1 and we have MA, Ded not ?

* * *

Moved from pgs.9-10

If we look at spaces with clopen basis, it seems easier to force such spaces. To enable the reader to read one proof the similar parts of the proofs are repeated.

6.1 Theorem. *Assume*

- (A) $\lambda > \kappa > \theta > \sigma \geq \aleph_0$ and $\kappa = \kappa^{<\kappa}, \kappa > \theta^*$
- (B) $\mathcal{A} \subseteq [\lambda]^\theta$ and $A_1 \neq A_2 \in \mathcal{A} \Rightarrow |A_1 \cap A_2| < \sigma$;
- (C) if $F : \lambda \rightarrow [\lambda]^{\leq \kappa}$ then for some $A \in \mathcal{A}$ is F -free, i.e. $\alpha \neq \beta \in A \Rightarrow \alpha \in F(\beta)$
- (D) Y^* is a T_3 topological space with set of points θ and with a clopen basis $\mathcal{B} = \{b_i : i < \theta^*\}$
- (E) if $Y \subseteq Y^*$ has cardinality $< \sigma$ then Y is closed.

THEN for some κ -complete κ^+ -c.c. forcing notion P in V^P there is X^* such that:

- (a) X^* is a T_3 topological space even with a clopen basis of size $|\mathcal{A}| + \theta^*$ with λ points (so its compactification is also compact and the other properties remain, also the Boolean Algebra of clopen sets is a free Boolean Algebra so the space is $\kappa 2$)
- (b) $X^* \rightarrow (Y^*)_{< \text{cf}(\theta)}^1$, i.e. if $X^* = \bigcup_{i < i(*)} X_i, i(*) < \text{cf}(\theta)$, then some closed subspace Y of Y^* homeomorphic to Y^* , is included in some single X_i .

* * *

Moved from pgs.11-14

Proof of g.4. Let $\mathcal{A} = \{A_\zeta : \zeta < \lambda^*\}$.

We define P :

$p \in P$ has the form $p = (u, v, v_*, \bar{w}) = (u^p, v^p, v_*^p, \bar{w}^p)$ such that:

- (α) $u \in [\lambda]^{<\kappa}$
- (β) $v_* \subseteq v \in [\lambda^*]^{<\kappa}$
- (γ) $\bar{w} = \langle w_{\zeta,i} : \zeta \in v \text{ and } i < \theta^* \rangle$
- (δ) $w_{\zeta,i} \subseteq u$
- (ε) $\zeta \in v_* \Rightarrow A_\zeta \subseteq u$
- (ζ) if $\zeta \in v$ and $i < \theta^*$ and $\xi \in v_* \setminus \{\zeta\}$ then $w_{\zeta,i} \cap A_\xi \in [A_\xi]^{<\sigma}$.

\oplus If $\zeta \in \lambda^* \setminus v$ we stipulate $w_{\zeta,i} = \emptyset$.

The order is: $p \leq q$ iff $u^p \subseteq u^q, v^p \subseteq v^q, v_*^p = v_*^q \cap v^q$ and

$$w_{\zeta,i}^p = w_{\zeta,i}^q \cap u^p.$$

Clearly

(*)₁ P is a partial order.

Our intention is to have X^* as follows: set of points λ
the clopen basis is generated by $\{u_{\zeta,i} : \zeta < \lambda^*, i < \theta^*\}$ where

$$u_{\zeta,i}[G_P] = \cup \{w_{\zeta,i}^p : p \in G_P, \zeta \in v^p\}.$$

No need to prove Hausdorff as otherwise we just need to identify $\alpha, \beta < \lambda$ if
($\forall \zeta, i$)($\alpha \in u_{\zeta,i} \equiv \beta \in u_{\zeta,i}$) but we will still do it.

(*)₂ P is κ -complete, in fact if $\langle p_\varepsilon : \varepsilon < \delta \rangle$ is increasing in P and $\delta < \kappa$ a
limit ordinal then $p = \bigcup_{\varepsilon < \delta} p_\varepsilon$ is an upper bound where $u^p = \bigcup_{\varepsilon < \delta} u^{p_\varepsilon}, v^p =$

$$\bigcup_{\varepsilon < \delta} v^{p_\varepsilon}, v_*^p = \bigcup_{\varepsilon < \delta} v_*^{p_\varepsilon} \text{ and } w_{\zeta,i}^p = \bigcup \{w_{\zeta,i}^{p_\varepsilon} : \varepsilon \text{ satisfies } \zeta \in v^{p_\varepsilon}, \varepsilon < \delta\}$$

[why? straight]

(*)₃ $P' = \{p : \text{if } \zeta < \lambda^*, |A_\zeta \cap u^p| \geq \sigma \text{ then } \zeta \in v_\zeta\}$
[why? for any $p \in P$ we define by induction on $\varepsilon < \sigma^+$, $p_\varepsilon \in P$ increasingly
continuous with ε . Let $p_\sigma = p$, if p_ε is defined let

$$v^{p_{\varepsilon+1}} = \{\zeta < \lambda^* : \zeta \in v^{p_\varepsilon} \text{ or } (A_\zeta \cap u^{p_\varepsilon}) \geq \sigma\}$$

$$v_*^{p_{\varepsilon+1}} = v_*^{p_\varepsilon}$$

$$u^{p_{\varepsilon+1}} = u^{p_\varepsilon} \cup \bigcup \{A_\zeta : \zeta \in v_{\varepsilon+1}\}$$

$w_{\zeta,i}^{p_{\varepsilon+1}}$ is: $w_{\zeta,i}^{p_\varepsilon}$ if $\zeta \in v^{p_\varepsilon}$ and \emptyset if $i \in v^{p_{\varepsilon+1}} \setminus v^{p_\varepsilon}$.

Clearly $p_\varepsilon \leq p_{\varepsilon+1} \in P$. Now $p_{\sigma^+} = \bigcup_{\varepsilon < \sigma^+} p_\varepsilon$ is as required.]

(*)₄ P satisfies the κ^+ -c.c.c.

[why? also easy. Let $p_j \in P$ for $j < \kappa^+$, without loss of generality $p_j \in P'$ for $j < \kappa^+$. By the Δ -system lemma for some unbounded $S \subseteq \kappa^+$ and $v^\otimes \in [\lambda^*]^{<\kappa}$, $u^\otimes \in [\lambda]^{<\kappa}$ we have: $j \in S \Rightarrow v^\otimes \subseteq v^{p_j}$ & $u^\otimes \subseteq u^{p_j}$ and $\langle v^{p_j} \setminus v^\otimes : j \in S \rangle$ are pairwise disjoint and $\langle u^{p_j} \setminus u^\otimes : j \in S \rangle$ are pairwise disjoint. Without loss of generality $\text{otp}(v^{p_j})$, $\text{otp}(u^{p_j})$ are constant and any two are isomorphic over v^\otimes, u^\otimes .

Now for $j_1, j_2 \in S$, p_{j_1}, p_{j_2} are compatible.]

(*)₅ $\{p : \alpha \in u^p \text{ and } \zeta \in v^p\}$ is dense open for each $\alpha < \lambda$

[why? if $p \in P$ let us define $q : u^q = u^p \cup \{\alpha\}$, $v^q = v^p \cup \{\zeta\}$, $v_*^q = v_*^p$ and $w_{\zeta,i}^q$ is $w_{\zeta,i}^p$ is well defined and empty otherwise.]

Now we come to the main point

(*)₆ in V^P , if $i(*) < \text{cf}(\theta)$, $X^* = \bigcup_{i < i(*)} X_i$ then some closed $Y \subseteq X^*$ is homeomorphic to Y^* .

Why? Toward contradiction assume $p^* \in P$ and $p^* \Vdash_P \langle \dot{X}_i : i < i(*) \rangle$ is a counterexample to (*)₅. Without loss of generality $p^* \Vdash_P \langle \dot{X}_i : i < i(*) \rangle$ is a partition of X^* , i.e. of λ .

For each $\alpha < \lambda$ let $\langle p_{\alpha,j}, i_{\alpha,j} : j < \kappa \rangle$ be such that:

- (i) $\langle p_{\alpha,j} : j < \kappa \rangle$ is a maximal antichain of P above p^*
- (ii) $p_{\alpha,j} \Vdash_P \text{"}\alpha \in \dot{X}_{i_{\alpha,j}}\text{"}$
- (iii) $p_{\alpha,j} \in P'$ (and $p^* \leq p_{\alpha,j}$).

Now choose a function F , $\text{Dom}(F) = \lambda$ and $F(\alpha)$ is $\kappa \cup \{u^{p_{\alpha,j}} : j < \kappa\}$.

So we can find $\zeta < \lambda^*$ such that:

if $\alpha \neq \beta$ are from $A_{\zeta(*)}$ then $\alpha \notin F(\beta)$.

Let $A_{\zeta(*)} = \{\beta_\varepsilon : \varepsilon < \theta\}$ with no repetitions. Now we shall choose by induction on $\varepsilon \leq \theta$, p_ε and if $\varepsilon < \theta$ also $j_\varepsilon < \kappa$ such that:

$$\begin{aligned}
 (a) \quad p_\varepsilon &\in P \text{ and } u^{p_\varepsilon} = u^{p^*} \cup \bigcup_{\varepsilon(1) < \varepsilon} u^{p_{\varepsilon(1)}, j_{\varepsilon(1)}} \\
 v^{p_\varepsilon} &= v^{p^*} \cup \bigcup_{\varepsilon(1) < \varepsilon} v^{p_{\beta_{\varepsilon(1)}}, j_{\varepsilon(1)}} \\
 v_*^{p_\varepsilon} &= v_*^{p^*} \cup \bigcup_{\varepsilon(1) < \varepsilon} v_*^{p_{\beta_{\varepsilon(1)}}, j_{\varepsilon(1)}} \\
 w_{\zeta,i}^{p_\varepsilon} &= w_{\zeta,i}^{p^*} \cup \bigcup_{\varepsilon(1) < \varepsilon} w^{p_{\beta_{\varepsilon(1)}}, j_{\varepsilon(1)}}
 \end{aligned}$$

(remember the convention \oplus)

(b) $j_\varepsilon = \text{Min}\{j < \kappa : p_{\beta_\varepsilon, j} \text{ is compatible with } p_\varepsilon\}$.

No problem to carry the definition (in limit ε take union, j_ε is well defined by clause (i) above). Let $i^* < i(*)$ be such that $Z = \{\varepsilon < \theta : i_{\beta_\varepsilon, j_\varepsilon} = i^*\}$ has cardinality θ . Note: $\zeta(*) \notin v^{p_{\beta_\varepsilon, j_\varepsilon}}$ as $A_{\zeta(*)} \cap F(\beta_\varepsilon)$ is a singleton so $|A_\zeta \cap u^{p_{\beta_\varepsilon, j_\varepsilon}}| \leq 1, p_{\beta_\varepsilon, j_\varepsilon} \in P'$. Now we define p :

$$u^p = u^{p_\theta}$$

$$v^p = v^{p_\theta} \cup \{\zeta(*)\}$$

$$v_*^p = v_*^{p_\theta} \cup \{\zeta(*)\}$$

$w_{\zeta, i}$ is:

$$(c) \ w_{\zeta, i}^{p_\theta} \text{ if } \zeta \in v^{p_\theta}$$

$$(d) \ \{\beta_\varepsilon : \varepsilon \in Z \text{ and } \text{otp}(Z \cap \varepsilon) \in b_i\} \text{ if } \zeta = \zeta(*).$$

Let $g : \theta \rightarrow \lambda$ be $g(\xi) = \beta_\varepsilon$ where $\xi < \theta, \varepsilon \in Z, \text{otp}(Z \cap \varepsilon) = \xi$. Now $p \geq p^*$ and p forces that:

- (α) if $b \in \mathcal{B}$ then for some open set U of $X, X \cap \{\beta_\varepsilon : \varepsilon \in Z\} = \{g(\varepsilon) : \varepsilon \in b\}$
[why? as $b = b_i$ for some i and p forces $w_{\zeta, i} \cap \{\beta_\varepsilon : \varepsilon \in Z\} = \{g(\varepsilon) : \varepsilon \in b_i\}$]
- (β) if $i < \theta^*, u_{\zeta(*), i} \cap \{\beta_\varepsilon : \varepsilon \in Z\}$ is of the form above
[why? clear]
- (γ) if $i < \theta^*, \zeta \in \lambda^* \setminus \{\zeta(*)\}$ then $u_{\zeta, i} \cap \{\beta_\varepsilon : \varepsilon \in Z\}$ has cardinality $< \sigma$ hence $g^{-1}(u_{\zeta, i} \cap \{\beta_\varepsilon : \varepsilon \in Z\})$ is a clopen subset of Y^*
[why? the first phrase as $\zeta(*) \in v_*^p$ and clause (ζ) in the definition of P ; the second follows]
- (δ) $\{\beta_\varepsilon : \varepsilon \in Z\}$ is a closed set in X
[why? let $\beta \in \lambda \setminus \{\beta_\varepsilon : \varepsilon \in Z\}, p \leq q \in P$, choose $\xi \in \lambda^* \setminus v^q$ and define q^+

$$v^{q^+} = v^q \cup \{\xi\}$$

$$v_*^{q^+} = v_*^q$$

$$u^{q^+} = u^q$$

$$w_{\zeta, i}^{q^+} \text{ is } w_{\zeta, i}^q \text{ if } \zeta \in v^q \text{ and is } \{\beta\} \text{ if } \zeta = \xi.]$$

(*)₇ in V^P , if $u_1 \subseteq u_2 \in [\lambda]^{<\sigma}$ then for some ζ, i we have

$$w_{\zeta, i} \cap u_2 = u_1$$

[why? let $p_0 \in P$ force $u_1 \subseteq u_2$ form a counterexample, as P is κ -complete some $p_1 \geq p_0$ forces $u_1 = u_2$ and $p_1 = P'$.

Let $\zeta(*) \in \lambda^* \setminus v^{p_1}$ and define $q \in P$:

$$u^q = u^{p_1} \cup u_2$$

$$v^q = v^p \cup \{\zeta\}$$

$$v_*^q = v_*^p$$

$$w_{\zeta,i}^q \text{ is: } w_{\zeta,i}^p \text{ if } \zeta \in v^q, u_1 \text{ if } \zeta = \zeta(*).$$

Now check.

Together all is done.]

→

scite{g.4} undefined

□?

g.8 Concluding Remark. 1) As in 1.6(1) we can allow $\kappa = \theta^*$.

22/12/97

1) in 3.2 without $\beth_2 = (2^{\aleph_0})^+$: more involved forcing: $\mathcal{U}^p \in [\lambda]^{2^{\aleph_0}}$ but we give only countable information (or $< 2^{\aleph_0}$?)

2) To get GCH? Try local forcing? (decision by u.t.?)

3) start with

4) 31/12/97

Question: in 4.4 try without (d), (e), (f) for every $f : \mathcal{U} \rightarrow X, \mathcal{U} \in J^+$?

Question: Replace \mathcal{F} by a family of functions?

We will prove a more detailed result. The analysis below is somewhat closed to the λ -sets from [Sh 161].

Definition. We define simultaneously by induction on $\lambda > \mu$ what is a partial (λ, μ) -index system.

1) A partial μ -index system Γ is a pair $(S, \bar{\lambda}) = (S^\Gamma, \bar{\lambda}^\Gamma)$ such that:

- (a) $\Gamma \subseteq {}^\omega \text{Ord}$
- (b) Γ is closed under initial segments,
- (c) $<> \in \Gamma$
- (d) $\bar{\lambda} = \langle \lambda_\eta : \eta \in S \rangle$ and $\lambda_\eta \geq \mu$
- (e) for each $\eta \in S$ for some $\alpha = \alpha(\eta, \Gamma) < \text{cf}(\lambda_\eta)$ we have

$$\eta^\wedge \langle \beta \rangle \in \Gamma \text{ iff } \beta < \alpha$$

(f)(α) if λ_η is a limit cardinal $> \mu$ then $\langle \lambda_{\eta^\wedge \langle \beta \rangle} : \beta < \alpha(\eta, \Gamma) \rangle$ is strictly increasing with limit μ

(β) if λ_η is a successor cardinal $> \mu$ then $\lambda_\eta = \lambda_{\eta^\wedge \langle \beta \rangle}^+$ for $\beta < \alpha(\eta, \Gamma)$

- (γ) if $\lambda_\eta = \mu$, then $\alpha(\eta, \Gamma) = 0$, i.e. η is Δ -maximal in S (so $\eta \triangleleft \nu \in S \Rightarrow \lambda_\eta > \lambda_\nu \geq \mu$)
- (g) the set $\{\eta \in S : \lambda_\eta > \mu \text{ but } \alpha(\eta, \Gamma) < \lambda_\eta\}$ has the form $\{\nu \restriction \ell : \ell < \ell(*)\}$, where $\nu = \text{Max}(S)$ is the maximal member of S in the lexicographic order and $\ell^* \leq \ell g(\nu)$, let $\nu \restriction \ell^*$ be called $\nu(\Gamma)$ [if $\ell^* = 0$ we stipulate $\nu(\Gamma) = \langle \rangle^-$ and $\eta \in \Gamma \Rightarrow \neg(\eta \triangleleft \nu)$].

Definition. 1) A full μ -index system Γ is a partial μ -index system such that $\lambda_\eta^\Gamma > \mu \Rightarrow \alpha(\eta, \Gamma) = \lambda_\eta^\Gamma$.

2) For $\Gamma = (S, \Gamma)$ a partial μ -index system, for $\eta \in S^\Gamma$ let $\Gamma^{[\eta]} = \langle S^{[\eta]}, \bar{\lambda}^{<\eta>} \rangle$ where

$$S^{<\eta>} = \{\nu : \eta \hat{~} \nu \in S\}$$

$$\bar{\lambda}^{<\eta>} = \langle \lambda_{\eta \hat{~} <\nu>} : \nu \in S^{<\eta>} \rangle.$$

We write also $\Gamma^{<\eta>} = (S^{\Gamma, <\eta>}, \bar{\lambda}^{\Gamma, <\eta>})$.

3) $\eta^+ = \nu$ if $\ell g(\eta) = \ell + 1, 1 = \ell g(\nu), \eta \restriction \ell = \nu \restriction \ell$ and $(\eta(\ell) + 1 = \nu(\ell))$.

Fact: If $\Gamma = (S, \bar{\lambda})$ a partial μ -index system we say \bar{N} is Γ -decomposition (in $\mathcal{H}(\chi)$)

- (a) $\bar{N} = \langle N_\eta : \eta \in \Gamma \rangle$
- (b) $N_\eta \prec (\mathcal{H}(\chi), \in, <_\chi^*)$
- (c) N_η has cardinality λ_η

Assignment: 1) GCH tail forcing?

2) 4.13? write a complete proof also larger μ so revise 1.2.

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